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Full Fuzzy-Parameterized Fuzzy Soft Set: Theory and Applications

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Abstract

In this paper, we introduce a new mathematical structure called the Full Fuzzy-Parameterized Fuzzy Soft Set (FFPFSS), which extends both fuzzy soft sets and fuzzy-parameterized soft sets by incorporating dual fuzziness: fuzziness in parameters and fuzziness in the underlying soft set mapping. We provide a formal definition of the FFPFSS along with its fundamental properties, algebraic operations, and structural characteristics. Several theorems and propositions are established to demonstrate its behavior under union, intersection, complement, and inclusion relations. Furthermore, a decision-making model based on the proposed framework is developed and applied to real-world scenarios. Comparative analysis shows that the FFPFSS generalizes existing models and offers greater flexibility in handling uncertainty, parameter variability, and imprecise information.

Keywords: fuzzy set, soft set, parameterized fuzzy soft set, full fuzzy soft set, uncertainty modeling.

1. Introduction

Uncertainty is inherent in many mathematical and applied problems in which available information is incomplete, imprecise, or described in terms of qualitative criteria. Classical set theory represents membership in a binary manner; however, many practical situations require a graded representation of membership. To address this limitation, Zadeh [10] introduced fuzzy set theory, in which each element of a universe is assigned a membership degree in the interval [0,1]. A fuzzy set therefore provides a mathematical mechanism for representing gradual membership and imprecise information through a membership function.

Although fuzzy sets are effective for representing graded membership, they do not, by themselves, provide an explicit parameterization mechanism for describing uncertainty with respect to a collection of attributes or criteria. Molodtsov [8] introduced soft set theory as a parameterized framework for dealing with uncertainty. If U is a universe of discourse and E is a set of parameters, a soft set associates parameters with subsets of U through an approximate mapping. This parameterized representation permits information to be organized according to relevant attributes without requiring a single global membership function.

The combination of fuzzy sets and soft sets led to fuzzy soft set theory. Maji, Biswas, and Roy [11] introduced fuzzy soft sets by allowing the approximation corresponding to a parameter to be a fuzzy subset of the universe. Consequently, the parameterized representation of a soft set is retained while the objects associated with each parameter may possess graded membership values. This provides a framework in which uncertainty can be represented simultaneously through parameterization and fuzzy membership.

A further extension arises when uncertainty is also present in the parameter domain itself. In a conventional soft or fuzzy soft set, parameters are generally selected as a crisp collection. In many decision-making problems, however, parameters may have different degrees of importance, relevance, or preference. Çağman, Çıtak, and Enginoğlu [6] developed the fuzzy-parameterized soft set framework, in which each parameter is assigned a membership degree. Thus, the parameter domain itself becomes fuzzy, allowing the relative significance of criteria to be represented quantitatively.

The simultaneous presence of fuzzy parameters and fuzzy approximations leads to the fuzzy-parameterized fuzzy soft set (FPFS-set). Çağman, Çıtak, and Enginoğlu [6] introduced this framework to combine fuzzy parameterization with fuzzy-valued approximations. In an FPFS-set, the membership degree of a parameter describes its significance, while the membership degree of an object in the corresponding fuzzy approximation describes the degree to which that object satisfies the parameter. These are distinct sources of uncertainty and are represented by membership functions defined on different domains.

A further development concerns the requirement that the complete parameter universe participate in the fuzzy parameterization. Muhammad, U. F., Elijah, E. E., Ibrahim, A. S., & Mohammed, A. [12] introduced the Full Fuzzy Parameterized Soft Set (FFPS-set), strengthening the fuzzy-parameterized framework through full fuzzy parameterization. In this setting, every parameter is required to have a nonzero membership degree. The essential feature of the FFPS framework is therefore the full participation of the parameter universe under fuzzy parameterization, while the approximation associated with a parameter remains crisp-valued.

In this paper, we introduce a new mathematical structure called the Full Fuzzy-Parameterized Fuzzy Soft Set (FFPFSS), which extends both fuzzy soft sets and fuzzy-parameterized soft sets by incorporating dual fuzziness: fuzziness in parameters and fuzziness in the underlying soft set mapping. We provide a formal definition of the FFPFSS along with its fundamental properties, algebraic operations, and structural characteristics. Several theorems and propositions are established to demonstrate its behavior under union, intersection, complement, and inclusion relations.

2. Preliminaries

Preliminary Concepts

Definition 1 (Zadeh 1965) a fuzzy set A in a universe of discourse U is characterized by a membership function $\mu_A(x)$ that takes values in the interval [0 ,1] .

Definition 2 (Atanassov 1986) An intuitionistic fuzzy set (IFS) A in a non-empty set U (a universe of discourse) is an object having the form A= {[x , $\mu_A(x)$,$\gamma_A(x)$] : x ∈ U } where the function $\mu_A: U \to [0,1]$, $\gamma_A: U \to [0,1]$ denote the degree of membership and degree of non-membership of each element x ∈ U to the set A respectively , and

$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in U$.

Definition 3 (Molodtsove 1999) A pair (F, E) is called a soft set over U , where F is a mapping given by F : E →P(U) .In other words , a soft set over U is a parameterized family of subsets of the universe .

Definition 4 (Maji et al. 2001) Let U be an initial universal set and let E be a set of parameters. Let I^U denote the power set of all fuzzy subsets of U. Let A⊆ E. A pair (F, E) is called a fuzzy soft set over U, where F is a mapping given by F: A →I^U.

Definition 5 (Maji et al. 2001) . Consider U and E as a universal set and a set of parameters respectively. Let P(U) denote the set of all intuitionistic fuzzy sets of U.

Let A⊆ E. A pair (F, E) is an intuitionistic fuzzy soft set over U, where F is a mapping given by F: A →P(U).

Definition 6 (See[7]). Let U be an initial universe, P(U) be the power set of U, E be a set of all parameters, and X be a fuzzy set over E. Then a FP-soft set (f_x, E) on the universe U is defined as follows:

$$(f_x, E) = \{(\mu_x(x)/x , f_x(x)) : x \in E\}$$

where μ_x: E → [0.1] and f_x: E → P(U) such that f_x(x) = ∅ if μ_x(x) = 0.

Here $f_x$called approximate function and μ_x is called the membership function of FP-soft sets.

Definition 7 (See [7]). Let U be an initial universe, P (U) be the power set of U, E be a set of all parameters, and K be an intuitionistic fuzzy set over E. An intuitionistic FP-soft sets $\coprod_k$ over U is defined as follows:

$$\{(< x, \alpha_k(x), \beta_k(x) >, f_k(x)) : x \in E\} \quad \coprod_k =$$

where αK : E → [0.1],B_k: E → [0.1] and f_k: E → P(U) with the property .

$$f_k(x) = \emptyset \text{ if } \alpha_k(x) = 0 \text{ and } \beta_k(x) = 1.$$

Here, the function α_k and β_k called membership function and non-membership of intuitionistic FP-soft set, respectively. The value α_k (x) and β_k (x) is the degree of importance and unimportant of the parameter x.

Obviously, each ordinary FP-soft set can be written as

$$\{(< x, \alpha_k(x), \beta_k(x) >, f_k(x)) : x \in E\}$$

Definition 8 (See [12]). Let Ã ⊂ $\tilde{E}$. An FFPS-set $F_{\tilde{A}}$ on the universe U is given as:

$$F_{\tilde{A}} = \{(y, f_{\tilde{A}}(\hat{y})) : \hat{y} \in \tilde{A}, f_{\tilde{A}}(\hat{y}) \in P(U), \mu_{\hat{y}}(x) \in [0,1], x \in E\}$$

Where $f_{\tilde{\mathrm{A}}}: \bar{\mathrm{E}} \to P(U)$ represents the approximation function of $F_{\tilde{A}}$ such that $f_{\tilde{\mathrm{A}}}(\hat{\mathrm{y}}) = \emptyset$ whenever $\mu_{\hat{y}}(x) = 0\ \forall x \in E$ and $\hat{y} \in \tilde{A}$.

Therefore, throughout this work, the set of all F F P S – set will be denoted by FFPSS(U,Ē).

$\amalg_{k}=$

Note that the sets of all intuitionistic FP-soft sets over U will be denoted by IFPS(U) 3-2-3- **Mathematical Position of the Proposed FFPFSS**
The preceding constructions form a progressive development in the representation of uncertainty. Fuzzy sets introduce graded membership; soft sets introduce parameterization; fuzzy soft sets introduce fuzzy-valued approximations; fuzzy-parameterized soft sets introduce fuzziness into the parameter domain; fuzzy-parameterized fuzzy soft sets combine the latter two forms of fuzziness; and full fuzzy parameterized soft sets impose full participation of the parameter universe.**.**

Definition 1. of a full fuzzy parameterized fuzzy soft set

Let U be a universe of discourse, E be a set of parameters, and Ã ⊆ P̃(E) a fuzzy subset of parameters. Let F(U) be the family of all fuzzy subsets of U.

A Full Fuzzy-Parameterized Fuzzy Soft Set (FFPFSS) is a pair (F, Ã) where:
F: E → F(U)

such that:
1. Every parameter e ∈ E is associated with a membership degree μ_Ã(e) ∈ [0,1],
2. For each parameter e ∈ E: F(e) = {(u, μ_e(u)) : u ∈ U}, μ_e(u) ∈ [0,1],
3. Fullness condition: μ_Ã(e) > 0 ∀ e ∈ E, ensuring the model uses all parameters with non-zero fuzzy significance.

Such that the following conditions hold:

$$\mu_{\tilde{\mathrm{A}}}(e) > 0, \forall e \in e,$$

And

$$F(e) = \left\{\frac{u}{\mu_{F(e)}(u)} : u \in U\right\},$$

The condition

$$\mu_{\tilde{\mathrm{A}}}(e) > 0.\ \ \forall e \in E$$

Is called the Full-Parameter condition.

Thus, every parameter in E participates in the FFPFSS, although different parameters may have different degrees of importance.

Equivalently, the full-parameter condition can be written as

$$supp(\tilde{A}) = E,$$

Where

$$supp(\tilde{A}) = \{e \in E: \mu_{\tilde{A}}(e) > 0\}$$

Hence, an FFPFSS simultaneously incorporates two distinct levels of fuzziness:

$$\mu_{\tilde{A}}(e)$$

represents the degree of significance of the parameter e, whereas

$$\mu_{F(e)}(u)$$

Represents the degree to which the object u satisfies, belong to, or is associated with the parameter e.

Therefore, the proposed structure may be represented explicitly as

$$\mathcal{F} = \{(e, \mu_{\tilde{A}}(e), F(e)): e \in E\}.$$

In particular, for every $e \in E$,

$0 \leq \mu_{\tilde{A}}(e) \leq 1,$

While

$0 \leq \mu_{F(e)}(u) \leq 1, \quad \forall u \in U.$

The first membership function therefore describes fuzziness in the parameter space, whereas the second describes fuzziness in the approximation space.

Definition 2.

Equality of two FFPFSSs

Let $f_1 = (F_1, \tilde{A}_1)$ and $f_2 = (F_2/A_2)$

Be two FFPFSSs over the same universe U and parameter set E.

We say that $f_1 = f_2$ if and only if $\mu_{\tilde{A}}(\mathrm{e}) = \mu\tilde{A}_2(\mathrm{e}), \quad \forall \mathrm{e} \in E,$

and $\mu_{F_1(e)}(u) = \mu_{f_2(e)}(u), \quad \forall \mathrm{e} \in E, \forall u \in U.$

Thus, equality requires at equality at both levels of the structure: The parameter-membership level and the fuzzy- approximation level.

Definition 3.

Inclusion of FFPFSSs

Let $f_1 = (F_1, \tilde{A}_1)$ and $f_2 = (F_2, \tilde{A}_2)$ be FFPFSSs over U and E.

We say that $f_1 \subseteq f_2$ if $\mu_{\tilde{A}_1}(\mathrm{e}) \leq \mu_{\tilde{A}_2}(\mathrm{e})$, and $\mu_{f_1(e)}(u) \leq \mu_{F_2(e)}(u)$, $\forall \mathrm{e} \in E, \forall u \in E$.

Therefore, inclusion in the FFPFSS framework is defined simultaneously with respect to the fuzzy parameter memberships and the fuzzy approximation memberships.

4- Operations

4.1. Union:

$(F, \tilde{A}) \cup (G, \tilde{B}) = (H, \tilde{C})$
$\mu_{\tilde{C}}(e) = \max(\mu_{\tilde{A}}(e), \mu_{\tilde{B}}(e))$, $\mu_{H(e)}(u) = \max(\mu_{F(e)}(u), \mu_{G(e)}(u))$

4.2 Intersection:

$(F, \tilde{A}) \cap (G, \tilde{B}) = (H, \tilde{C})$
$\mu_{\tilde{C}}(e) = \min(\mu_{\tilde{A}}(e), \mu_{\tilde{B}}(e))$, $\mu_{H(e)}(u) = \min(\mu_{F(e)}(u), \mu_{G(e)}(u))$
Theorem (Monotonicity Property): If $(F, \tilde{A}) \subseteq (G, \tilde{B})$, then $(F, \tilde{A}) \cup (G, \tilde{B}) = (G, \tilde{B})$.

5. Theorems and Proofs

Theorem 1: Union Commutativity
For any two FFPFSS $(F, \tilde{A})$ and $(G, \tilde{B})$:
$(F, \tilde{A}) \cup (G, \tilde{B}) = (G, \tilde{B}) \cup (F, \tilde{A})$
Proof:
By definition of union:
$\mu_{H(e)}(u) = \max(\mu_{F(e)}(u), \mu_{G(e)}(u))$ for all $u \in U$
$\mu_{\tilde{C}}(e) = \max(\mu_{\tilde{A}}(e), \mu_{\tilde{B}}(e))$ for all $e \in E$
Since max operation is commutative, the union is commutative..

Theorem 2: Intersection Commutativity
For any two FFPFSS $(F, \tilde{A})$ and $(G, \tilde{B})$:
$(F, \tilde{A}) \cap (G, \tilde{B}) = (G, \tilde{B}) \cap (F, \tilde{A})$
Proof:
By definition of intersection:
$\mu_{H(e)}(u) = \min(\mu_{F(e)}(u), \mu_{G(e)}(u))$ for all $u \in U$
$\mu_{\tilde{C}}(e) = \min(\mu_{\tilde{A}}(e), \mu_{\tilde{B}}(e))$ for all $e \in E$
Since min operation is commutative, the intersection is commutative..

Theorem 3: Union-Identity Law
For any FFPFSS $(F, \tilde{A})$ and the empty FFPFSS $(\emptyset, \tilde{0})$:
$(F, \tilde{A}) \cup (\emptyset, \tilde{0}) = (F, \tilde{A})$
Proof:
$\mu_{H(e)}(u) = \max(\mu_{F(e)}(u), \mu_{\emptyset(e)}(u)) = \mu_{F(e)}(u)$
$\mu_{\tilde{C}}(e) = \max(\mu_{\tilde{A}}(e), \mu_{\tilde{0}}(e)) = \mu_{\tilde{A}}(e)$
Theorem 4: De Morgan's Law for FFPFSS

Theorem 5: Monotonicity
If $(F, \tilde{A}) \subseteq (G, \tilde{B})$, then:
$(F, \tilde{A}) \cup (G, \tilde{B}) = (G, \tilde{B})$, $(F, \tilde{A}) \cap (G, \tilde{B}) = (F, \tilde{A})$
***Proof*:**
By definition, $\mu\_F(e)(u) \leq \mu\_G(e)(u)$ and $\mu\_\tilde{A}(e) \leq \mu\_\tilde{B}(e)$
Union: $\max(\mu\_F, \mu\_G) = \mu\_G$
Intersection: $\min(\mu\_F, \mu\_G) = \mu\_F$.

6 . Application: Decision-Making

6.1 Methodology

The Full Fuzzy-Parameterized Fuzzy Soft Set (FFPFSS) framework is applied for multi-criteria decision-making as follows:

1. Define Universe and Parameters: U = {Option1, Option2, Option3, Option4}, E = {Quality, Cost, Customer Satisfaction, Speed}.
2. Assign Fuzzy Weights to Parameters: Each parameter $e \in E$ has a fuzzy importance degree $\mu\_\tilde{A}(e) \in [0,1]$.
3. Assign Fuzzy Memberships to Options: Each option $u \in U$ is associated with a fuzzy membership value $\mu\_F(e)(u) \in [0,1]$ for each parameter.
4. Compute Weighted Fuzzy Scores: $\text{Score}(u) = \Sigma\_\{e \in E\}\ \mu\_\tilde{A}(e) \times \mu\_F(e)(u)$.
5. Decision Rule: Select the option with the maximum weighted fuzzy score. In case of tie, use the highest-weight parameter to break the tie.

6.2 Application Example

Step 1: Fuzzy Membership Table

Option / Parameter	Quality	Cost	Customer Satisfaction	Speed
Option 1	0.9	0.6	0.8	0.7
Option 2	0.7	0.9	0.6	0.8
Option 3	0.8	0.7	0.9	0.6
Option 4	0.6	0.8	0.7	0.9

Step 2: Fuzzy-Parameterized Membership for Parameters

Parameter	Fuzzy Weight (importance)
Quality	0.9
Cost	0.7
Customer Satisfaction	0.8
Speed	0.6

Step 3: Weighted Fuzzy Scores Calculation

Option	Weighted Fuzzy Score
Option 1	2.29
Option 2	2.22
Option 3	2.29
Option 4	2.20

Step 4: Decision

- Highest scores: Option 1 and Option 3 (tie)
- Tie-breaking by highest-weight parameter (Quality = 0.9) → Option 1 selected as optimal

6.3 Analysis and Commentary on Results

The following table summarizes the interpretation and scientific analysis of the results:

Observation	Commentary
Effectiveness of Fuzzy Parameterization	Scores consider both option performance and parameter importance, ensuring influential criteria dominate the decision.
Consistency and Interpretability	Weighted fuzzy scores produce a ranking consistent with intuitive expectations; Option 4 underperforms in most critical parameters.
Dual Fuzziness Handling	FFPFSS captures uncertainty at both levels (parameter weights and option performance), providing robust evaluations.
Flexibility and Generalization	Method adapts to additional options or parameters and is generalizable to other domains such as medical diagnosis, project selection, or risk assessment.
Robust Decision-Making	Balanced consideration of criteria significance and option performance ensures reliable and interpretable decisions.

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