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Approximate Analytic Solution of The Fractional Rosenau -Hyman Equation Using Fractional Power Series Method

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Abstract

In this study, fractional Rosenau-Hynam equations is considered. We implement analytical technique, the for solving this equation using Fractional power series method. The fractional derivatives are described in the Caputo sense. The method in applied mathematics can be used as alternative method for obtaining analytic and approximate solutions for fractional Rosenau-Hynam equations. the solution takes the form of a series with easily computable components. The present method performs well in terms of efficiency and simplicity, which is one of the useful and the results reveal that the method is very effective and simple.

Keyword: fractional power series: Caputo fractional derivative: The Fractional Rosenau-Hyman Equation.

1 Introduction

Recent advances of fractional differential equations are stimulated by new examples of applications in fluid mechanics, viscoelasticity, mathematical biology, electrochemistry, and physics. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [5] can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional differential equations for seepage ow in porous media are suggested in [7], and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [1]. Fractional differential equations also have studied and successfully solved

Most nonlinear differential equations are usually arising from mathematical modeling of many physical systems Many phenomena in applied sciences and engineering can be described successfully by using fractional order differential equation [4][5]. But fractional order differential equations are usually hard to solve analytically and exact solution are rather difficult to be obtained. So, it is very important to find efficient methods for solving these equations. Various researchers have introduced new method in the literature. fractional power series method is simpler and more effective.

In the present paper, we will use fractional power series method (FPSM) [8] be applied for solving fractional Rosenau-Haynam equation which written as:

$$D_t^\alpha u = uD_{xxx}(u) + uD_x(u) + 3D_x(u)D_{xx}(u) \qquad t > 0 \tag{1}$$

where $u = u(x,t)$ and D^a is the Caputo derivative of order α and $0 < \alpha \leq 1$ subject to the initial condition

$$u(x,0) = \frac{-8}{3} c \cos^2\left(\frac{x}{4}\right), \tag{2}$$

Equation (1) can be rewritten as follows:

$$D_t^\alpha u = u(u)_{xxx} + u(u)_x + 3u_x u_{xx} \tag{3}$$

2 Basic definitions

Definition 1.([3]) A power series representation of the form

$$\sum_{n=0}^{\infty} c_n\ (t - t_0) = ({}_n\mathrm{t}\mathrm{c}_0 + \mathrm{c}_1 (t - t_0)^\alpha + \mathrm{c}_2 (\mathrm{t} - \mathrm{t}_0)^{(2\alpha)} + \ldots \tag{4}$$

Where $0 \leq m - 1 < \alpha \leq 1$, $m \in N^+$ and $t \geq t_0$ is called a fractional power series (FPS) about t_0, where t is a variable and c_n are the coeffients of the series.

Definition 2. ([4]) The fractional derivative of $f(x)$ in Caputo sense is defined as

$$D^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s)ds, \qquad (5)$$

For $m-1 < \alpha \leq 1$, $m \in N^+$, $x > 0$.

In addition, the following theorem would be necessary in order to use fractional power series method.

Theorem 1. ([6]) Suppose that the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ hasradius of convergence $R > 0$, If $f(t)$ is function defined by:

$f(t) = \sum_{n=1}^{\infty} c_n t^{n\alpha}$ on $0 \leq t < R$; then for $m-1 < \alpha \leq 1$ and $0 \leq t < R$, we have

$$D^{\alpha}f(t) = \sum_{n=1}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha} \qquad (6)$$

3 Application

In this section, we use the FPSM to solve the fractional Rosenau-Haynam equation

3.1 Application 1

we suppose that the solution of (1) takes the following form

$$u(x,t) = \sum_{n=0}^{\infty} u_n t^{n\alpha} = (u_0 + u_1 t^{\alpha} + u_2 t^{2\alpha} + \cdots) \qquad (7)$$

By Theorem 1, on the one hand, we have

$$D^{\alpha}u(x,t) = \sum_{n=1}^{\infty} u_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha} \qquad (8)$$

On the other hand

$$u_x = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} u_n t^{n\alpha} = \frac{\partial}{\partial x} u_0 + \frac{\partial}{\partial x} u_1 t^{\alpha} + \frac{\partial}{\partial x} u_2 t^{2\alpha} + \cdots \qquad (9)$$

$$u_{xx}=\sum_{n=0}^{\infty}\frac{\partial^2}{\partial x^2}u_n t^{n\alpha}=\frac{\partial^2}{\partial x^2}u_0+\frac{\partial^2}{\partial x^2}u_1 t^{\alpha}+\frac{\partial^2}{\partial x^2}u_2 t^{2\alpha}+\cdots \tag{10}$$

$$u_{xxx}=\sum_{n=0}^{\infty}\frac{\partial^3}{\partial x^3}u_n t^{n\alpha}=\frac{\partial^3}{\partial x^3}u_0+\frac{\partial^3}{\partial x^3}u_1 t^{\alpha}+\frac{\partial^3}{\partial x^3}u_2 t^{2\alpha}+\cdots \tag{11}$$

Substituting (8), (9), (10) and (11) into (1), and comparing the coefficients of t^{α} in the both sides, we get

$$\begin{aligned}\sum_{n=1}^{\infty}u_n\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}t^{(n-1)\alpha}
&=\left(\sum_{n=0}^{\infty}u_n t^{n\alpha}\right)\left(\sum_{n=0}^{\infty}\frac{\partial^3}{\partial x^3}u_n t^{n\alpha}\right)+\left(\sum_{n=0}^{\infty}u_n t^{n\alpha}\right)\left(\sum_{n=0}^{\infty}\frac{\partial}{\partial x}u_n t^{n\alpha}\right)\\
&\quad+3\left(\sum_{n=0}^{\infty}\frac{\partial}{\partial x}u_n t^{n\alpha}\right)\left(\sum_{n=0}^{\infty}\frac{\partial^2}{\partial x^2}u_n t^{n\alpha}\right)\end{aligned}$$

$$\begin{aligned}u_1\Gamma(\alpha+1)t^{\alpha}+u_2\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}t^{2\alpha}+u_3\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}t^{3\alpha}+\cdots
&=(u_0+u_1t^{\alpha}+u_2t^{2\alpha}+u_3t^{3\alpha}+\cdots)\left(\frac{\partial^3}{\partial x^3}u_0+\frac{\partial^3}{\partial x^3}u_1t^{\alpha}\right.\\
&\quad\left.+\frac{\partial^3}{\partial x^3}u_2t^{2\alpha}+\cdots\right)\\
&\quad+(u_0+u_1t^{\alpha}+u_2t^{2\alpha}+u_3t^{3\alpha}+\cdots)\left(\frac{\partial}{\partial x}u_0+\frac{\partial}{\partial x}u_1t^{\alpha}+\frac{\partial}{\partial x}u_2t^{2\alpha}\right.\\
&\quad\left.+\cdots\right)\\
&\quad+3\left(\frac{\partial}{\partial x}u_0+\frac{\partial}{\partial x}u_1t^{\alpha}+\frac{\partial}{\partial x}u_2t^{2\alpha}+\cdots\right)\left(\frac{\partial^2}{\partial x^2}u_0+\frac{\partial^2}{\partial x^2}u_1t^{\alpha}\right.\\
&\quad\left.+\frac{\partial^2}{\partial x^2}u_2t^{2\alpha}+\cdots\right)\end{aligned}$$

$$u_1\Gamma(\alpha+1)=u_0\frac{\partial^3}{\partial x^3}u_0+u_0\frac{\partial}{\partial x}u_0+3\left(\frac{\partial}{\partial x}u_0\right)\left(\frac{\partial^2}{\partial x^2}u_0\right)+\cdots$$

$$u_2\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}=u_1\frac{\partial^3}{\partial x^3}u_1+u_1\frac{\partial}{\partial x}u_1+3\left(\frac{\partial}{\partial x}u_1\right)\left(\frac{\partial^2}{\partial x^2}u_1\right)+\cdots$$

using initial condition $u(x,0)=\frac{-8}{3}c\cos^2\left(\frac{x}{4}\right)$, we have $u_0(x,0) = \frac{-8}{3} c \cos^2\left(\frac{x}{4}\right)$,

Next we determine the $u_n (n = 1,2,3,\ldots)$

Therefore, we obtain the approximate solution of the fractional Rosenau-Haynam equation

$$u_1 = \frac{1}{\Gamma(\alpha+1)}\left[-\frac{2}{3}c^2\sin\left(\frac{x}{2}\right) - \frac{4}{3}c\left[1+\cos\left(\frac{x}{2}\right)+\cdots\right]\right]$$

$$u_2 = \frac{1}{\Gamma(2\alpha+1)}\left[-\frac{4}{3}c^2\cos\left(\frac{x}{2}\right) + \left[\frac{1}{6}c^2+\frac{4}{3}c\right]\left[1+\cos\left(\frac{x}{2}\right)\right] \backslash -\frac{1}{6}c^2 + \frac{2}{3}c^2\sin\left(\frac{x}{2}\right)+\cdots\right]$$

$$u_3 = \frac{1}{\Gamma(3\alpha+1)}\left[\frac{7}{6}c^3\cos\left(\frac{x}{2}\right) - \frac{96}{36}c\cos\left(\frac{x}{2}\right) - + \frac{48}{36}c^2\sin\left(\frac{x}{2}\right) - \frac{2}{3}\cos\left(\frac{x}{2}\right) + 6c^2\sin\left(\frac{x}{2}\right)\right] - \frac{2}{3}c^2\sin\left(\frac{x}{2}\right)+\cdots],$$

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$$

$$u(x,t) = \frac{1}{\Gamma(\alpha+1)}\left[-\frac{2}{3}c^2\sin\left(\frac{x}{2}\right) - \frac{4}{3}c\left[1+\cos\left(\frac{x}{2}\right)+\cdots\right] + \frac{1}{\Gamma(2\alpha+1)}\left[-\frac{4}{3}c^2\cos\left(\frac{x}{2}\right) + \left[\frac{1}{6}c^2+\frac{4}{3}c\right]\left[1+\cos\left(\frac{x}{2}\right)\right] - \frac{1}{6}c^2 + \frac{2}{3}c^2\sin\left(\frac{x}{2}\right)+\cdots\right] + \frac{1}{\Gamma(3\alpha+1)}\left[\frac{7}{6}c^3\cos\left(\frac{x}{2}\right) - \frac{96}{36}c\cos\left(\frac{x}{2}\right) - + \frac{48}{36}c^2\sin\left(\frac{x}{2}\right) - \frac{2}{3}\cos\left(\frac{x}{2}\right) + 6c^2\sin\left(\frac{x}{2}\right)\right] - \frac{2}{3}c^2\sin\left(\frac{x}{2}\right)+\cdots$$

4 Conclusion

we have used FPSM to solve some linear or nonlinear fractional differential equations. Compared to the other method, the FPSM is more simple, direct and effective.

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