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Characteristic Fuzzy subgroups and Abelian Fuzzy subgroups

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Abstract

In this paper we present some basic concepts of fuzzy set theory. With properties of normal fuzzy subgroups. Fuzzy analogs of some group theoretic concepts such as cosets, conjugate fuzzy subgroups, quotient groups. The order of fuzzy subgroup. Also, we introduce analogs of some group-theoretic concepts such as characteristic subgroups. We prove that if μ is a fuzzy subgroup of a group G such that the fuzzy index of μ is the smallest prime dividing the order of G, then μ is a normal fuzzy subgroup.
the work presented in [4], [5], [6], [8], [12], [13], [14],[16], [17], [18].

Key words: Characteristic Fuzzy subgroups**,** abelian Fuzzy subgroups, normal fuzzy subgroup, G-fuzzy subgroup.

الملخص:

في هذه, الورقة نقدم بعض المفاهيم الأساسية لنظرية المجموعات الضبابية. مع خصائص الزمر الضبابية العادية, النظائر الضبابية لبعض نظرية الزمر مثل المجموعات المشاركة, الزمر الضبابية المترافقة, مجموعات القسمة, رتبة الزمرة الضبابية، كما نقدم نظائر لبعض مفاهيم نظرية الزمر مثل الزمرة المميزة ونثبت أنه إذا كانت u زمرة ضبابية من الزمرة G بحيث يكون الدليل الضبابي ل u هو أصغر عدد أولي يقسم رتبة G. فأن u زمرة ضبابية عادية. العمل المقدم في [4], [5], [6], [8], [12], [13], [14],[16], [17], [18].

الكلمات المفتاحية:
الزمرة الجزئية الضبابية المميزة, الزمرة الجزئية الضبابية التبديلية, الزمرة الجزئية الطبيعية, G- زمرة جزئية ضبابية.

Introduction

The concept of fuzzy set theory was first introduced by Zadeh [9] and Rosenfeld on fuzzy subgroup of a group in [2]. This theory has been applied for alot of areas such as the theory of semi-group and the theory of groups [2]. Fuzzy cosets and fuzzy normal subgroups of group *G* have been studied in [1], [12]. In [6]. We write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum. We let $\mathbb{N}$ denote the set of positive integers.

Throughout this study *G* denotes a group and I denotes the unit closed interval [0, 1]. *A* fuzzy subset of X is a function from X into [0, 1]. The set of all fuzzy subset of *X is* called the fuzzy power set of X and is denoted by $\mathcal{FP}(X)$. The identity of *G* will be denoted by *e.*

2- Preliminaries

In this section, we recall some definition and theorems which we needed in this paper, groups, subgroups, normal subgroups and a homomorphism and Isomorphism. The materials in this section are taken from [1], [3], [10].

Definition(2.1)

If H is a subgroup of G, $a \in G$, then $Ha = \{ha: h \in H\}$ is called the right coset of H in G, and $aH = \{ah: h \in H\}$ is called the left coset of H in G.
If G is commutative, then of course $aH = Ha$. Observe that
$eH = H = He$ and that $a = ae \in aH$ and $a = ea \in Ha$.

Definition(2.2)

If H is a subgroup of G, the index of H in G is the number of distinct right cosets of H in G, and is denoted by $[G: H]$.

Definition (2.3)

A subgroup N of G is said to be *a normal subgroup* of G if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$. and is denoted by $N \trianglelefteq G$.

Remark(2.1)

If G is Abelian, then any subgroup N is normal, $g \in G$, $n \in N$
$gng^{-1} = gg^{-1}n = en = n \in N$.

Lemma(2.1)

The subgroup N of G is a normal subgroup of G if and only if every left coset of N is a right coset of N in G.

Definition (2.4)

Let $(G, *)$ *and* (H, o) be two groups.

a mapping $f: G \to H$ is called a homomorphism if and only if
$f(a * b) = f(a) \, o \, f(b)$ *for all* $a, b \in G$.

Definition (2.5)

If $f: G \to H$ is a homomorphism, then

(1) If f is one-to-one, then f is called monomorphism.
(2) If f is onto, then f is called an epimorphism .
(3) If f is one-to-one and onto, then f is called an isomorphism, and we say the group G is an isomorphic to the group H, and denoted by $G \cong H$.
(4) If f from a group G to itself, then f is called an endomorphism of G.
(5) If f is an isomorphism from a group G to itself, then f is called an authomorphism of G.

Definition (2.6)

Let $f: G \to H$ be homomorphism.
The kernel of f (denoted by *ker f*) is defined by: $\ker f = \{x \in G: f(x) = e_H\}$.

Theorem (2.1)

if $f: G \to H$ is a homomorphism, then $\ker f \trianglelefteq G$.

Theorem (2.2)

Let $f: G \to H$ be a homomorphism. The f is one-to-one if and only if $\ker f = \{e_G\}$.

3- Fuzzy subgroups

In order to define the notion of fuzzy subgroup and to examine its properties, we introduce some operations on a fuzzy subset of a group *G* in terms of the group operation, define some basic concepts of a fuzzy
a lgebra, the fuzzy product $vo\mu$, a fuzzy subgroup of G. The materials in this chapter are taken from the following references [2], [7], [11], [13] [15], [19].

Definition (3.1)

Let $\mu \in \mathcal{FP}(G)$. Then the set $\{\mu(x): x \in X\}$ is called the image of μ and is denoted by $\mu(X)$ or $Im(\mu)$. The set $\{x: x \in X, \mu(x) > 0\}$, is called the ***support*** of μ and is denoted by μ^*. In particular, μ is called a ***finite fuzzy subset*** of X if μ^* is a finite set, and an infinite fuzzy subset otherwise.

Definition (3.2)

Let *G* be a group. A fuzzy subset μ of a group *G* is called a fuzzy subgroup of the group *G* if

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for every $x, y \in G$ and

(ii) $\mu(x^{-1}) = \mu(x)$ for every $x \in G$.

Theorem (3.1)

Let $\mu \in \mathcal{F}(G)$. Then

(i) $\mu_* = \{x \in G: \mu(x) = \mu(e)\} \leq G.$

(ii) $\mu^* = Support\ of\ \mu = \{x \in G: \mu(x) > 0\}$, then $\mu_*\ and\ \mu^*$ are subgroups of a group G.

Remark (3.1)

Denote by $\mathcal{F}(G)$. the set of all fuzzy subgroups of G. If $\mu \in \mathcal{F}(G)$. we let
it can be easily verified that if μ is a fuzzy subgroup of a group *G*, then $\mu(x) \leq \mu(e)$ for every $x \in G$.

Definition (3.3)

Let X be a groupoid , and $v, \mu \in \mathcal{FP}(x)$. Then the product $\nu o \mu$ is defined as follows:

$$(\nu o \mu)(z) = \begin{cases} \sup\{\min(\nu(x), \mu(y))\} & for\ x, y \in X, z = x.y \\ o, & otherwise(z \neq x.y) \end{cases}$$

It is clear that $vo\mu \in \mathcal{FP}(x)$.

Example (3.1)

Let $X = (Z^+ \cup \{0\}, +)$, and $v, \mu \in \mathcal{FP}(x)$ for any $x, y \in Z^+ \cup \{0\}$.
A ssume that $z = 3 \in Z^+ \cup \{0\}$, $\mu(0) = 0.2$, $\mu(1) = 0.5$, $\mu(2) = 0.6$ and $\mu(3) = 0.4$, also $\nu(0) = 0.5$, $\nu(1) = 0.25$, $\nu(2) = 0.9$ and $\nu(3) = 0.75$, then $(\nu o \mu)(3) = o.5$.
If $z = 1 \in Z^+ \cup \{0\}$, and $\mu(0) = 0.2$, $\mu(1) = 0.5$,
Also, $\nu(0) = 0.5$, $\nu(1) = o.25$, then $(\nu o \mu)(1) = 0.2$.
But if $z = 0 \in Z^+ \cup \{0\}$, then $(\nu o \mu)(0) = 0$.

Remark (3.2)

Let "." be a binary operation in *X*, if x_t is a fuzzy point and μ is any fuzzy subset of *X* and $x_t \leq \mu$, then we write $x_t \in \mu$. Note that $x_t \in \mu$ if and only if $x_t \in \mu$ is a level subset of μ. If x_r and y_s are fuzzy points, then
$x_r\, o\, y_s = (x,y)_{\min\{r,s\}}$.

4-properties of Normal Subgroups

In this section, we study Fuzzy Cosets, and Fuzzy normal subgroups in G, conjugate fuzzy subgroup and quotient group. The materials in this section are taken from the following references [4], [11], [12], [13], [14].

Definition (4.1)

Let μ be a fuzzy subgroup of a group *G*. For any $a \in G$, aμ defined by $(a\mu)x = \mu(a^{-1}x)$ for every $x \in G$ is called the fuzzy
coset of the group *G* determined by *a* and *μ.*

Theorem (4.1)

Let $\mu \in \mathcal{FP}(G)$. Then the following assertions are equivalent:

(1) $\mu(yx) = \mu(xy)\ \forall\, x,\ y \in G$; in this case, μ is called an Abelian fuzzy subset of *G*.

(2) $\mu(xyx^{-1}) = \mu(y)\ \forall\, x,\ y \in G$.

(3) $\mu(xyx^{-1}) \geq \mu(y)\ \forall\ x,\ y \in G$.

(4) $\mu(xyx^{-1}) \leq \mu(y)\ \forall\ x,\ y \in G$.

(5) $\mu o v = v o \mu\ \ \forall v \in \mathcal{FP}(G)$.

Definition (4.2)

Let *G* be a group. *A fuzzy subgroup μ* of *G* is called normal if
$\mu(x) = \mu(y^{-1}xy)$ for all $x, y \in G$.

Theorem (4.2)

Let $\mu \in \mathcal{NF}(G)$. Then μ_* and μ^* are normal subgroups of *G*.

Proof

Since $\mu \in \mathcal{F}(G)$, it follows from Theorem (4.5) that μ_* *and* μ^* are subgroups of *G*. Let $x \in G$ *and* $y \in \mu_*$. Since μ satisfies condition (2) of Theorem (5.2), we have $\mu(xyx^{-1}) = \mu(y) = \mu(e)$ and thus $xyx^{-1} \in \mu_*$. Hence μ_* is a normal subgroup of *G.* Let $x \in G$ *and* $y \in \mu^*$. Since μ satisfies condition (2) of Theorem (5.2), it follows tha $\mu(xyx^{-1}) = \mu(y) > 0$ *and so* $xyx^{-1} \in \mu^*$. Therefore, μ^* is a normal subgroup of *G*.

Definition (4.3)

Let $\mu \in \mathcal{F}(G)$. Then μ is called a normal fuzzy subgroup of G if it is an Abelian fuzzy subset of G. *Let* $\mathcal{NF}(\mathrm{G})$ denote the set of all If $\mu, v \in \mathcal{F}(G)$ and there exists $u \in G$ such that $\mu(x) = v(uxu^{-1})\forall x$
Then μ *and* v are called conjugate fuzzy subgroup (with respect to u)
and we write, $\mu = v^u$, where $v^u(x) = v(uxu^{-1})\forall x \in G$.

Remarks (4.1)

(1) Clearly, 1_G and $1_{\{e\}}$ are normal fuzzy subgroups of *G*.
(2) If *G* is a commutative group, then every fuzzy subgroup of *G* is normal.
(3) A fuzzy subgroup μ of *G* is normal if and only if $\mu = \mu^g\ \forall\, g \in G$.
normal fuzzy subgroups of G.

Definition (4.4)

Let μ be a fuzzy subgroup of a finite group G. Then the cardinality of G/μ is called the index of μ in G, written $[G: \mu]$.

Theorem (4.3)

Let μ ∈ $\mathcal{NF}(G)$. $Set\ G/\mu = \{x\mu \ / \ x \in G\}$. Then the following assertions hold:

(1) $(x\mu)o(y\mu) = (xy)\mu \ \forall x, y \in G$.

(2) $(G/\mu, O)$ is a group.

(3) $G/\mu \cong G/\mu_*$.

(4) $Let\ \mu^{(*)} \in \mathcal{FP}(G/\mu)$ be defined by $\mu^{(*)}(x\mu) = \mu(x) \ \forall \ x \in G$. Then $\mu^{(*)} \in \mathcal{NF}(G/\mu)$.

The group G/μ defined in Theorem (5.7) is called the quotient group (or factor group) of G relative to the normal fuzzy subgroup

5- Characteristic Fuzzy subgroups and A belian Fuzzy subgroups

We prove a number of results a bout fuzzy groups involving the concepts of fuzzy cosets and fuzzy normal subgroups. These results are analogs standard results from group theory, conjugate fuzzy subgroups. Also, we introduce analogs of some group-theoretic concepts such as characteristic subgroups. We prove that if μ is a fuzzy subgroup of a group G such that the fuzzy index of μ is the smallest prime dividing the order of G, then μ is a normal fuzzy subgroup.

the work presented in [1], [6], [12], [14], [16], [17], [18].

Definition (5.1)

Let μ be a fuzzy subgroup of a group G and θ a function from G into itself. Define the fuzzy subset μ^θ of G by

$$\forall \ x \ \in G, \quad \mu^\theta(x) = \mu\left(x^\theta\right), \ where \ \ x^\theta = \theta(x)$$

Definition (5.2)

a fuzzy subgroup μ on a group K is called a fuzzy Characteristic subgroups of G if $\mu^\theta(x) = \mu(x)$ for every automorphism θ of G and all $x \in G$.

Theorem (5.1)

Let μ be a fuzzy subgroup of a group G. Then the following assertions hold.

(1) If θ is a homomorphism of G into itself, then μ^θ is a fuzzy subgroup of G.

(2) If μ is a fuzzy characteristic subgroup of G, then μ is a normal.

Proof.

(1) $Let\ x, y \in G$. Then

$$\mu^\theta(xy) = \mu\big((xy)^\theta\big)$$
$$= \mu\left(x^\theta y^\theta\right)$$

Since θ is a homomorphism. Since μ is a fuzzy subgroup of G.

$$\mu\big(x^\theta y^\theta\big) \geq \mu\big(x^\theta\big) \wedge \mu\big(y^\theta\big)$$
$$= \mu^\theta(x) \wedge \mu^\theta(y)$$

Thus $\mu^\theta(xy) \geq \mu^\theta(x) \wedge \mu^\theta(y)$.

Also,

$$\mu^\theta(x^{-1}) = \mu\big((x^{-1})^\theta\big) = \mu\left(\left(x^\theta\right)^{-1}\right) = \mu\big(x^\theta\big) = \mu^\theta(x).$$

Hence μ^θ is a fuzzy subgroup of G.

(2) Let $x, y \in G$. To prove that μ is normal, we have to show that

$$\mu(xy) = \mu(yx).$$

Let θ be the function of G into itself defined by

$$\theta(z) = x^{-1}zx \quad \forall\, z \in G.$$

It is a standard result that θ is an automorphism of G (called the inner automorphism induced by x). Now since μ is a fuzzy characteristic subgroup of G, $\mu^{\theta} = \mu$. Thus

$$\begin{aligned}\mu(xy) &= \mu^{\theta}(xy)\\ &= \mu\left((xy)^{\theta}\right)\\ &= \mu(yx).\end{aligned}$$

Hence μ is normal.

Definition (5.3)

Let μ_1, μ_2 be two fuzzy subgroup of a group G. We say that μ_1 is conjugate to μ_2 if there exists $y \in G$ such that $\forall x \in G,\ \mu_1(x) = \mu_2(y^{-1}xy)$.

Remark (5.1)

if μ_1 *and* μ_2 are fuzzy subgroups of a group G such that μ_1 *is conjugate to* $\mu_{2,}$ then μ_1 *and* μ_2 are conjugate fuzzy subgroups of G as defined previously.

Lemma (5.1)

Let μ be a fuzzy subgroup of a finite group G. Let $K = \left\{x \in {}^{G}/_{x\mu} = e\mu\right\}$. Then K is a subgroup of G. In fact, $K = \mu_*$.

Theorem (5.2)

If μ is a fuzzy subgroup of a finite group G such that the fuzzy index of μ *is* p, the smallest prime dividing the order of G, then μ is a normal fuzzy subgroup of G.

Proof.

By theorem (3.1), μ_* is a subgroup of G. Further, by Lemma (5.2) And [18], μ_* has index p in G, that is, μ_* has p distinct (Left) cosets, say, $\{x_i\mu_* \,/\, 1 \le i \le p\}$. $\forall\, x \in G$, define the function π_x *of* ${}^{G}/_{\mu_*}$ into itself by $\forall g \in G, \pi_x(g\mu_*) = xg\mu_*$. Then π_x is a one-to-one Function of ${}^{G}/_{\mu_*}$ onto itself. Let $P(G) = \{\pi_x \,/\, x\epsilon G\}$. Now consider the permutation representation of G on the cosets of μ_* given by the function π of G onto $P(G)$, where $\pi(x) = \pi_x\ \forall x \in G$.

π is an isomorphism of G into the symmetric group Sym(p), since the index of μ_* in G is p. The kernel of the map π is the core of μ_*, written $Core(\mu_*)$, the intersection of all the conjugates $g^{-1}\mu * g,\ g \in G$.

By the fundamental theorem of homomorphism of group and using Lagrange's theorem, we have that the order of $G \,/\, Core(\mu_*)$ divides $p!$, Which is the order of Sym(p). Since ${}^{G}/_{\mu_*} \cong (G/Core(\mu_*))/(\mu_*/Core(\mu_*))$.

$$|(G/Core(\mu_*)| = \left|{}^{G}/_{\mu_*}\right| |\mu_* \,/\, Core(\mu_*)|.$$

Now the order of ${}^{G}/_{\mu_*}$ is p. Thus it follows that the order of $\mu_*/Core(\mu_*)$ divides $(p-1)!$. Now since the order of μ_* divides the order of G, we obtain that $\mu_* = Core(\mu_*)$, otherwise we get a contradiction of the fact that p is the smallest prime dividing the order of G. it follows that μ_* is normal subgroup of G. Now consider the quotient group $G/_{\mu_*}$. Since the order of $G/_{\mu_*}$ is Abelian. Hence $\forall x, y \in G$, we have

$$(x\mu_*)(y\mu_*) = (y\mu_*)(x\mu_*)$$
$$\Longrightarrow \quad xy\mu_* = yx\mu_*$$
$$\Longrightarrow \quad xy = yxh$$

For some $h \in \mu_*$. Hence

$$\begin{aligned}\mu(xy) &= \mu(yxh)\\ &\geq \mu(yx) \wedge \mu(h)\\ &\geq \mu(yx) \wedge \mu(e)\\ &= \mu(yx).\end{aligned}$$

Thus $\mu(xy) \geq \mu(yx)$.

Similarly, we obtain that

$$\mu(yx) \geq \mu(xy).$$

Hence $\mu(xy) = \mu(yx)$ $\forall x, y \in G$. Therefore, μ is normal.

Corollary (5.1)

IF μ is a fuzzy subgroup of G such that the fuzzy index of μ is 2, then μ is normal.

We now derive an analog of the following result from group theory which states that if is homomorphism of a group G into itself whose kernel is N, then θ induces a homomorphism from $G/_N$ into itself.

Theorem (5.3)

Let μ be a normal fuzzy subgroup of G and let θ be a homomorphism of G into itself which leaves invariant the subgroup μ_*. Then θ induces a homomorphism $\bar{\theta}$ of $G/_\mu$ into itself defined by $\bar{\theta}(x\mu) = \theta(x)\mu$ $\forall x \in G$.

Proof.

First we show that $\bar{\theta}$ is well defined. To show this, suppose x, $y \in G$ are such that

$$x\mu = y\mu.$$

Then it suffices to prove that $\theta(x)\mu = \theta(y)\mu$.

Since $x\mu = y\mu$. We have that $x\mu(x) = y\mu(x)$, $x\mu(y) = y\mu(y)$

$$\Longrightarrow \quad \mu(e) = \mu(y^{-1}x),\ \mu(x^{-1}y) = \mu(e)$$
$$\Longrightarrow \quad \mu(y^{-1}x) = \mu(x^{-1}y)$$
$$\Longrightarrow \quad y^{-1}x,\ x^{-1}y \in \mu_*.$$

Since by hypothesis $\theta(\mu_*) = \mu_*$, we get that $\theta(y^{-1}x)$ and $\theta(x^{-1}y)$ also belong to μ_*. Thus we have

$$\mu\big(\theta(y^{-1}x)\big) = \mu\big(\theta(x^{-1}y)\big) = \mu(e).$$

Now let $g \in G$. Then

$$\begin{aligned}\theta(x)\mu(g) &= \mu(\theta(x^{-1})g)\\ &= \mu(\theta(x^{-1})\theta(y)\theta(y^{-1})g)\\ &\geq \mu(\theta(x^{-1})\theta(y)) \wedge \mu(\theta(y^{-1})g)\\ &= \mu(\theta(x^{-1}y)) \wedge \theta(y)\,\mu(g)\\ &= \mu(e) \wedge \theta(y)\,\mu(g)\\ &= \theta(y)\mu(g).\end{aligned}$$

Thus

$$\theta(x)\mu(g) \geq \theta(y)\mu(g).$$

Similarly, we have that

$$\theta(x)\mu(g) \leq \theta(y)\mu(g).$$

Since $g \in\in G$ is arbitrary, we have

$$\theta(x)\mu = \theta(y)\mu.$$

Therefore, $\bar{\theta}$ is well defined. And we show that $\bar{\theta}$ is a homomorphism. Let $x, y \in G$. Since θ is a homomorphism, $\theta(xy) = \theta(x)\theta(y)$. Thus $\theta(xy) = \theta(x)\theta(y)\mu$. Hence $\bar{\theta}(xy\mu) = \theta(x)\mu \ o\ \theta(y)\mu$. Thus

$$\bar{\theta}(x\mu \ o\ y\mu) = \bar{\theta}(x\mu) \ o\ \bar{\theta}(y\mu).$$

Therefore, $\bar{\theta}$ is a homomorphism.

Corollary (5.2)

With the some hypothesis as in Theorem (5.3). the function θ is an automorphism of G if $\bar{\theta}$ is an automorphism and $\mu_* = \{e\}$.

Proof. Let $x,\ y \in G$ be such that $\theta(x) = \theta(y)$. Then is follows that $\theta(x)\mu = \theta(y)\mu$.

That is, $$\bar{\theta}(x\mu) = \bar{\theta}(y\mu).$$

Hence , $x\mu = y\mu$ since $\bar{\theta}$ is one-to-one by the hypothesis . Thus

$$x\mu(y) = y\mu(y),$$

which implies that

$$\mu(x^{-1}y) = \mu(e).$$

Therefore, $x^{-1}y \in \mu_*$ $and\ so\ x^{-1}y = e\ \ since\ \mu_* = \{e\}$ by the hypothesis. Thus $x = y$. $Hence\ \theta$ is one-to-one.

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