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Theoretical Analysis and Approximate Solutions of Two-Dimensional Nonlinear Fredholm Integral Equations Using Adomian polynomials (ADM) and MATLAB

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Abstract

In this paper, the existence and uniqueness of the solution for the two-dimensional non-linear Fredholm integral equation are established using the fixed-point theorem, incorporating Adomian's approximation and the Lipschitz condition. We derive the approximate analytical solution Using the Adomian polynomials (Adomian decomposition method(ADM)) and compare it directly with the exact solution using MATLAB. Furthermore, we present various cases solved both analytically and numerically, evaluating each against the exact solution. The findings demonstrate a rapid convergence rate for ADM, confirming its superior computational efficiency in Two-dimensions**.**

Keywords :Two-dimensional Fredholm integral equation , Adomian Decomposition Method, Approximate solution, Numerical solution, Absolute error.

التحليل النظري والحلول التقريبية للمعادلات التكاملية غير الخطية من نوع فردهولم ثنائية الأبعاد باستخدام كثيرة حدوديات ادوميان (ADM) وبرنامجMATLAB

الملخص

في هذا البحث، تم إثبات وجود ووحدانية الحل للمعادلة التكاملية غير الخطية من نوع فردهولم ثنائية الأبعاد باستخدام نظرية النقطة الثابتة، مع دمج تقريب أدوميان وشرط لِبسشتز. ولقد تم ايجاد الحل التحليلي التقريبي باستخدام طريقة أدوميان التكرارية باستخدام كثيرة حدوديات ادوميان (ADM) وتمت مقارنته مباشرة مع الحل الدقيق باستخدام برنامج MATLAB علاوة على ذلك، نُقدّم حالات مختلفة تم حلها تحليلياً وعددياً، مع المقارنة بالحل الدقيق. وتُظهر النتائج معدل تقارب سريع لطريقةADM ، مما يؤكد كفاءتها الحسابية الفائقة في البعد الثاني.

الكلمات المفتاحية: معادلة فردهولم التكاملية ثنائية الابعاد، طريقة أدوميان التفككية، الحل التقريبي، الحل العددي، الخطأ المطلق

1- Introduction

Integral equations are very important because they help us model many physical and engineering problems [8, 11, 12]. Among these, two-dimensional (2D) non-linear Fredholm integral equations have become popular because they can describe real-world phenomena that change across different spaces [1, 2, 6, 10]. However, finding the exact solutions for these equations is usually very difficult or even impossible, because the non-linear parts and the 2D domains make the math highly complex [3, 7, 12]. This is why developing practical and efficient approximation methods is a main focus in applied mathematics today. The general form non-linear Fredholm integral equation of the second kind given by

$$u(x,y) = f(x,y) + \lambda \int_a^b \int_c^d k(x,y;t,s)\varphi(t,s,u(t,s))dtds \qquad (1).$$

Here, $k(x,t;y,z)$ and $f(x,y)$ are known functions in the space $C([a,b] \times [c,d])$ *with* $\|.\|$ [10] , and they are called the kernel and the free term of the non-linear Fredholm integral equation, respectively. Also, $\varphi(t,s,u(t,s))$is known continuous function, while $u(x,y)$ is unknown

2- The Adomian decomposition method [1, 2, 3, 4, 7, 8,9,10]

Introduced by George Adomian in the 1980, the Adomian Decomposition Method (ADM) is a powerful analytical and semi-analytical technique widely utilized for solving complex functional equations. The pivotal innovation of this method lies in its elegant treatment of nonlinear terms, which are decomposed into a series of specially generated polynomials known as Adomian Polynomials. Consequently, when extended to two-dimensional nonlinear Fredholm integral equations of the second kind, ADM provides distinct computational advantages. It circumvents the need for discretization or linearization, offering a rapidly converging continuous solution in the form of an analytic series.

This method consists of decomposing the unknown function $u(x,y)$ of any equation into a sum of an infinite number of components defined by the series:

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) \qquad (2),$$

where all the components $u_n(x,y), n \geq 0$ can be computed is recursive manner. The non-linear function $\varphi(t,s,u(t,s))$ of Eq. (1) is presented by the adomian series

$$\varphi(t,s,u(t,s)) = u^n(t,s) = \sum_{n=0}^{\infty} A_n(t,s) \qquad (3\ )$$

But how do we actually find these Adomian polynomials A_n? To make it easy, we temporarily introduce a placeholder variable, This lets us rewrite our unknown function as a moving power series

$$u(\lambda) = \sum_{i=0}^{\infty} \lambda^i u_i$$

By taking successive derivatives of our nonlinear function with respect to this placeholder λ and then setting $\lambda = 0$ clear away the scaffolding we get a universal formula to generate each polynomial

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2,\dots$$

Applying this formula yields the first four polynomials

$$A_0 = \varphi(u_0)$$

$$A_1 = u_1 \varphi'(u_0)$$

$$A_2 = u_2 \varphi'(u_0) + \frac{1}{2} {u_1}^2 \varphi''(u_0)$$

$$A_3 = u_3 \varphi'(u_0) + u_1 u_2 \varphi''(u_0) + \frac{1}{6} {u_1}^3 \varphi'''(u_0)$$

…………………………

Substituting(2) and (3) in to (1) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = f(x,y) + \lambda \int_a^b \int_c^d k(x,y;t,s) \left(\sum\nolimits_{n=0}^{\infty} A_n(t,s) \right) dtds \qquad (4).$$

The recurred relation is used to determine the components

$$u_0(x,y) = A_0(x,y) = f(x,y)$$

$$u_1(x,y) = \lambda \int_a^b \int_c^d k(x,y;t,s) A_0(t,s) dtds$$

$$u_2(x,y) = \lambda \int_a^b \int_c^d k(x,y;t,s) A_1(t,s) dtds$$

$$u_3(x,y) = \lambda \int_a^b \int_c^d k(x,y;t,s) A_2(t,s) dtds$$

…………………………………………..

$$u_n(x,y) = \lambda \int_a^b \int_c^d k(x,y;t,s) A_{n-1}(t,s) dtds$$

It is possible to rewritten equation (4) as

$$\sum_{n=0}^{\infty} u_n(x,y) = u_0(x,y) + u_1(x,y) + u_2(x,y) + \cdots \qquad (5).$$

Where

$$\begin{cases} A_0(x,y) = f(x,y) \\ u_{n+1}(x,y) = \lambda \int_a^b \int_c^d k(x,y;t,s) A_n(t,s)\,dt\,ds \end{cases} \tag{6}$$

The solution of each component $u_0, u_1, u_2, \ldots$are absolutely find. As a result, the solution $u(x,y)$ of (1) in series from is obtained by using the decomposition (2).

3- The existence and uniqueness of the solution [2,5,6,10,11]

In this the section, we give proof for existence and uniqueness of solution for two-dimensional non-linear Fredholm integral equation (1) in the Banach space $C([a,b]\times[c,d])$ with $\|.\|$, under the following conditions:

a) The kernel $k(x,t,y,z)$ satisfies the condition

$$|\lambda| \left\{ \int_a^b \int_a^b \int_c^d \int_c^d |k(x,y;s,t)|^2 dxdydtds \right\}^{\frac{1}{2}} \leq M,$$

where M is a small enough constant.

b) The given function $f(x,y)$ and its partial derivatives with respect to x, y are continuous, and its norm in $C([a,b]\times[c,d])$ with $\|.\|$is defined as

$$\|f(x,y)\| = \left[\int_a^b \int_c^d |f(x,y)|^2 dxdy \right]^{\frac{1}{2}} = B$$

c) The unknown function $u(x,y)$ satisfies Lipschitz condition for the arguments x, y

$$\left\{ \int_a^b \int_c^d |\varphi(t,s,u(t,s)) - \varphi(t,s,w(t,s))|^2 \right\}^{\frac{1}{2}} \leq \varepsilon$$

if

$$\|\varphi(t,s,u(t,s)) - \varphi(t,s,w(t,s))\| \leq \delta(\varepsilon) \qquad 0 < \epsilon < 1$$

Theorem

Using the previous conditions, the solution of (1) is exist and unique , where $|\lambda| < \frac{1}{M}$.

Proof:

We will prove this theorem by another way which differs on [5]'s proof, which is considered a new method of proof. We will use Adomian decomposition method to prove this theorem. First, we will prove the exist solution of equation (1) , by we set the function $u_0(x,y) = f(x,y)$, then the other components constructing as follows

$$u_1(x,y) = \lambda \int_a^b \int_c^d k(x,y;t,s)\varphi(t,s,u_0(t,s))dtds$$

$$= \lambda \int_a^b \int_c^d k(x,y;t,s)f(t,s)dtds$$

$$u_2(x,y) = \lambda \int_a^b \int_c^d k(x,y;t,s)\varphi(t,s,u_1(t,s))dtds$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_n(x,y) = \lambda \int_a^b \int_c^d k(x,y;t,s)\varphi(t,s,u_{n-1}(t,s))dtds \ ; n \geq 1$$

Now, taking the norm of both sides produces

$$\|u_n(x,y)\| \leq \left\| \lambda \int_a^b \int_c^d k(x,y;t,s)\varphi(t,s,u_{n-1}(t,s))\,dtds \right\|$$

$$\leq |\lambda| \left\| \int_a^b \int_c^d k(x,y;t,s)\varphi(t,s,u_{n-1}(t,s))\,dtds \right\|$$

For $n = 1$, we have

$$\|u_1(x,y)\| \leq |\lambda| \left\| \int_a^b \int_c^d k(x,y;t,s)\varphi(t,s,u_0(t,s))\,dtds \right\|$$

then applying the Cauchy-Schwarz inequality, we get

$$\|u_1(x,y)\| \leq |\lambda| \left\{ \int_a^b \int_c^d \left| \int_a^b \int_c^d |k(x,y;t,s)\varphi(t,s,u_0(t,s)dtds| \right|^2 dxdy \right\}^{\frac{1}{2}}$$

$$\leq |\lambda| \left\{ \int_a^b \int_a^b \int_c^d \int_c^d |k(x,y;t,s)|^2 dxdydtds \right\}^{\frac{1}{2}} \|\varphi(t,s,u_0(t,s))\|$$

Now, by using conditions (a) and (b), we obtain

$$\|u_1(x,y)\| \leq |\lambda|MB = \sigma B$$

where $\sigma = |\lambda|M$, and for n=2

$$\|u_2(x,y)\| \leq |\lambda| \left\{ \int_a^b \int_c^d \left| \int_a^b \int_c^d |k(x,y;t,s)\varphi(t,s,u_1(t,s))dtds| \right|^2 dxdy \right\}^{\frac{1}{2}}$$

$$\leq |\lambda|M\|\varphi(t,s,u_1(t,s))\| \leq \sigma(\sigma B) = \sigma^2 B$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots.$$

$$\|u_n(x,y)\| \leq \sigma^n B$$

Where $|\lambda| < \frac{1}{M}$, then the sequence $u_n(x,y)$ uniformly convergent. Hence, we get

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$$

since each of $u_n(x,y)$ is continuous, convergent and represents the existence of the solution equation (1).

Second, we prove the solution of equation (1) is unique, assume $w(x,y)$ is another solution, hence

$$[u(x,y) - w(x,y)] = \lambda \int_a^b \int_c^d k(x,y;t,s)[\varphi(t,s,u(t,s)) - \varphi(t,s,w(t,s))]dtds$$

Now, taking the norm of both sides produces

$$\|u(x,y) - w(x,y)\| \leq \left\| \lambda \int_a^b \int_c^d k(x,y;t,s)[\varphi(t,s,u(t,s)) - \varphi(t,s,w(t,s))]dtds \right\|$$

Applying Cauchy-Schwarz inequality, we have

$\|u(x,y) - w(x,y)\|$

$$\leq |\lambda| \left\{ \int_a^b \int_c^d \left| \int_a^b \int_c^d |k(x,y;t,s)[\varphi(t,s,u(t,s)) - \varphi(t,s,w(t,s))]dtds| \right|^2 dxdt \right\}^{\frac{1}{2}}$$

$$\leq |\lambda| \left\{ \int_a^b \int_a^b \int_c^d \int_c^d |k(x,y;t,s)|^2 dxdydtds \right\}^{\frac{1}{2}} \left\{ \int_a^b \int_c^d |\varphi(t,s,u(t,s)) - \varphi(t,s,w(t,s))|^2 \, dtds \right\}^{\frac{1}{2}}$$

then using conditions (a) and (c), we obtain

$$\|u(x,y) - w(x,y)\| \leq |\lambda| M\varepsilon$$

since $|\lambda| < \frac{1}{M}$ and $0 < \varepsilon < 1$, then $u(x,y) = w(x,y)$.

Example1. consider two-dimensional nonlinear fredholm integral equation **[12]**

$$u(x,y) = x^2y^2 - \frac{1}{36}xy + \int_0^1 \int_0^1 (xyst)\,[u(s,t)]^2 dsdt$$

Using the Adomian Decomposition Method, we express the unknown solution $u(x,y)$ as an infinite series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$$

We also break down the nonlinear part using Adomian polynomials A_n Eq.of (6)

the initial component of the decomposition series is

$$A_0(x,y) = u_0(x,y) = x^2y^2 - \frac{1}{36}xy$$

By applying the Adomian Decomposition method recurrence relation ,

$$u_1(x,y) = \int_0^1 \int_0^1 (xyst) \left[s^2t^2 - \frac{1}{36}st \right]^2 dsdt$$

$$= \left(\frac{1}{36} - \frac{1}{450} + \frac{1}{20736} \right) xy$$

Next, the overall solution is assembled by adding the calculated components into the infinite decomposition series:

$$u(x,y) = u_0 + u_1 + u_1 + \cdots.$$

Substituting our explicit expressions into this series gives the multi-term approximation

$$u(x,y) = x^2y^2 - \frac{1}{36}xy + \left(\frac{1}{36} - \frac{1}{450} + \frac{1}{20736}\right)xy + \cdots \ldots$$

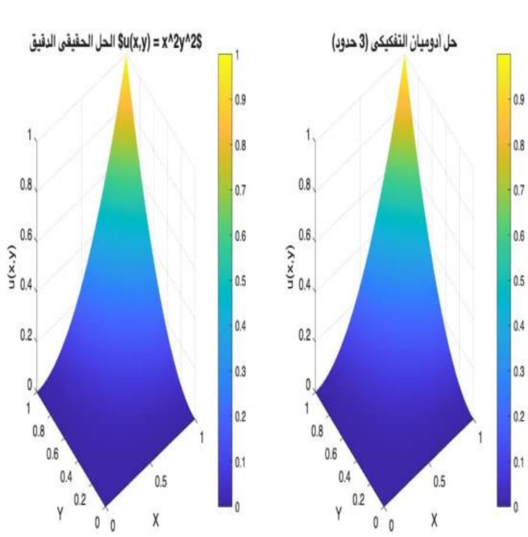
As n → ∞, the limit the term xy vanishes and transforms into the exact solution

$$u(x,y) = x^2y^2$$

Table 1. Numerical comparison between the exact and Adomian solutions.

x	y	Exact Solution	Adomian Solution	Absolute Error
0.25	0.25	0.003906	0.003893	1.34×10^{-5}
0.25	0.50	0.015625	0.015598	2.68×10^{-5}
0.25	0.75	0.035156	0.035116	4.02×10^{-5}
0.25	1.00	0.062500	0.062446	5.36×10^{-5}
0.50	0.50	0.062500	0.062446	5.36×10^{-5}
0.50	0.75	0.140625	0.140545	8.05×10^{-5}
0.50	1.00	0.250000	0.249893	1.07×10^{-4}
0.75	0.75	0.316406	0.316286	1.21×10^{-4}
0.75	1.00	0.562500	0.562339	1.61×10^{-4}
1.00	1.00	1.000000	0.999785	2.15×10^{-4}

Table 1



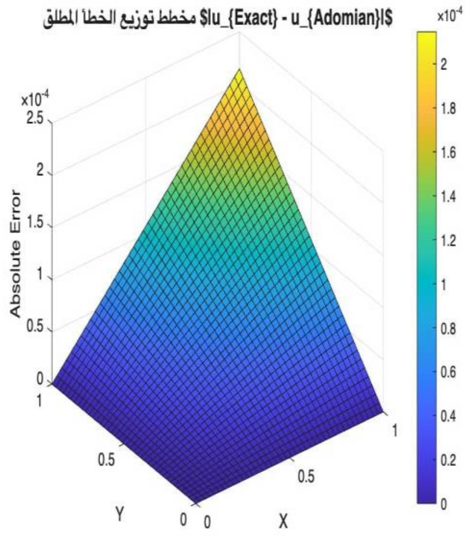


Figure.1 Figure .2

Figure 1. 3D surface plot comparison between the exact solution (left) and the Adomian decomposition method (right) for the original equation.

Figure 2. Absolute error distribution map for the ADM solution across the domain [0.25,1]

Result and discussion.1

Table1 presents a numerical comparison showing that the Adomian method yields results remarkably close to the exact solution. Note that boundary points where $x = 0$ or $y = 0$ are excluded from the discussion because both solutions equal exactly zero, making the absolute error a perfect zero.

For the remaining points, the method maintains exceptional accuracy. The error starts at a minimal 1.34×10^{-5} at $(0.25, 0.25)$ and reaches a tiny maximum of just 2.15×10^{-4} at $(1,1)$. This high precision, achieved using only the first few components, demonstrates how fast and reliable this method is without requiring significant computational effort.

Example.2 consider two-dimensional nonlinear fredholm integral equation

$$u(x,y) = x + y + \int_0^1 \int_0^1 (xy^2) \sin\big(u(s,t)\big)\, dsdt$$

$$u(x,y) = x + y + (xy^2) \int_0^1 \int_0^1 \sin\big(u(s,t)\big)\, dsdt$$

Using the Adomian Decomposition Method, we express the unknown solution $u(x,y)$ as an infinite series

$$u(x,y)=\sum_{n=0}^{\infty}u_n(x,y)$$

We also break down the nonlinear part $\sin(u)$ using Adomian polynomials A_n

$$\sin(u)=\sum_{n=0}^{\infty}A_n$$

The requrrence relation are

$$u_0(x,y)=x+y$$

$$u_{n+1}(x,y)=xy^2\int_0^1\int_0^1 A_n(s,t)dsdt$$

the Adomian polynomial are

$$A_0=\sin(u_0)$$

$$=\sin(s+t)$$

Substituting A_0 in to the relation

$$u_1(x,y)=xy^2\int_0^1\int_0^1 \sin(s+t)dsdt$$

$$=2\,sin(1)-sin(2)\approx 0.77364$$

Hence ,we get

$$u_1(x,y)=0.77364xy^2$$

We use $A_1=u_1\cos(u_0)$ inside the recurrence formula, we obtain

$$A_1=u_1\cos(u_0)=0.77364st^2\cos(s+t)$$

$$u_2(x,y)=0.77364xy^2\int_0^1\int_0^1 st^2cos(s+t)dsdt$$

$$u_2(x,y)=0.01861xy^2$$

$$u(x,y)=u_0+u_1+u_{2+}\dots\dots$$

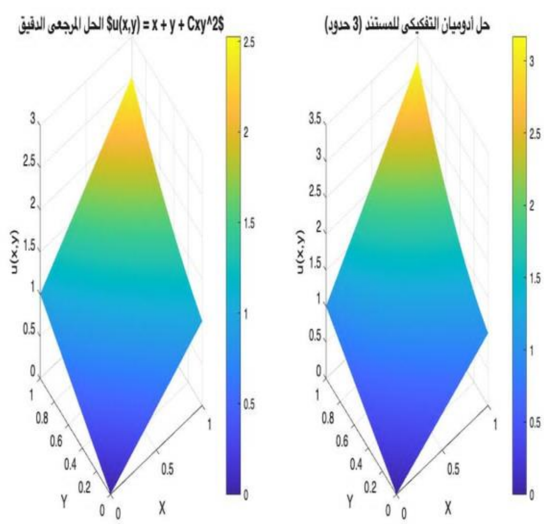
$$u(x,y)=x+y+0.77364\,xy^2+0.01861xy^2+\dots.$$

$$u(x,y)\cong x+y+0.79225xy^2$$

Table 2 : Numerical comparison between the exact and Adomian solutions

x	y	Exact Solution	Adomian Solution	Absolute Error
0.25	0.25	0.50821	0.51823	1.0017×10^{-2}
0.25	0.50	0.78285	0.82292	4.0070×10^{-2}
0.25	0.75	1.07390	1.16410	9.0157×10^{-2}
0.25	1.00	1.38140	1.54170	1.6028×10^{-1}
0.50	0.50	1.06570	1.14580	8.0140×10^{-2}
0.50	0.75	1.39780	1.57810	1.8031×10^{-1}
0.50	1.00	1.76280	2.08330	3.2056×10^{-1}
0.75	0.75	1.72170	1.99220	2.7047×10^{-1}
0.75	1.00	2.14420	2.62500	4.8084×10^{-1}
1.00	1.00	2.52550	3.16670	6.4112×10^{-1}

Table 2



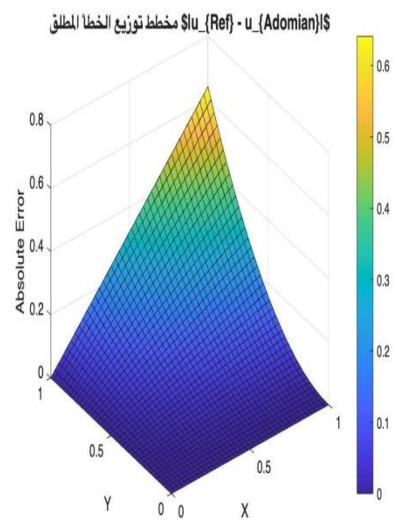


Figure.3

Figure.4

Figure 3. 3D surface plot comparison between the exact solution (left) and the 3-term Adomian decomposition method (right).

Figure 4**.** Absolute error distribution map for the 3-term ADM solution across the domain [0.25,1]

Results and discussion 2

Table2 The numerical results demonstrate the high accuracy of the Adomian decomposition method (ADM) for solving this nonlinear Fredholm integral equation. An excellent agreement is observed between the exact and approximate solutions at the coordinated points. Crucially, the absolute error decreases consistently as the variables (x, y) approach the origin, dropping from 10^{-1} to10^{-2} and eventually reaching a perfect zero at the boundaries. This steady decline confirms the rapid convergence and numerical stability of ADM without requiring significant computational effort

Conclusion

this study successfully validated the efficiency of the Adomian Decomposition Method (ADM) by using Adomian polynomials in solving the given integral equation by demonstrating its high accuracy and rapid convergence. Furthermore, the existence of the solution was rigorously established using the same proposed solution framework. The numerical results closely matched the exact solution with a negligible absolute error that decreased consistently across the domain. These outcomes confirm that the method is a powerful, stable, and reliable mathematical tool for obtaining highly precise approximate solutions without the need for complex transformations.

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