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Solving Singular Differential-Algebraic System Subject to Nonlocal Integral Boundary Conditions Using Generalized Inverse

Jamila Alhadi A Alhawil

Department of Mathematics, Abu-Issa Faculty of Education, University of Zawia, Libya

Corresponding Author's Email: j.alhawil@zu.edu.ly

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Abstract:

This paper investigates a novel class of first-order linear singular differential-algebraic systems subject to generalized multi-point nonlocal integral boundary conditions. By systematically utilizing the Moore-Penrose generalized inverse and orthogonal operators, the intractable singular system is decoupled and regularized into an equivalent tractable ordinary differential system. Furthermore, by incorporating distributed integral constraints over the entire domain, we derive a modified generalized algebraic shooting matrix. Rigorous existence and uniqueness criteria for the solution are established under standard tractability assumptions. Finally, an illustrative numerical example is provided to demonstrate the practical validity and efficiency of the proposed analytical framework.

Keywords: Singular differential systems, generalized inverse, integeral boundary conditions, Fundamental matrix, Characteristic operator.

1. Introduction:

Singular systems of ordinary differential equations, frequently formulated as differential-algebraic equations (DAEs), provide indispensable mathematical models for constrained dynamical processes across modern engineering, optimal control theory, and electrical network simulation [4, 6]. Unlike standard explicit differential equations, singular systems possess leading coefficient matrices that are non-invertible, leading to inherent mathematical complexities such as impulsive behaviors, inconsistent initial conditions, and potential non-uniqueness of solutions [4].

To address these analytical challenges, numerous reduction and regularization techniques have been developed in recent decades [4, 5]. Recent investigations by Rameshpagilla and Divya (2023) have significantly advanced the analysis of multi-point boundary value problems associated with generalized matrix differential systems, emphasizing the need for a robust explicit solution framework [2]. Furthermore, contemporary studies by Al-Smadi et al. (2025) and Zhang and Wang (2026) have highlighted the growing importance of hybrid boundary terms and nonlocal integral constraints in capturing hereditary and cumulative physical phenomena[1,3]. However, the rigorous formulation and exact solvability of singular systems coupled with nonlocal integral boundary conditions via generalized inverses remains an open and vital domain for theoretical enhancement.

Building upon foundational reduction techniques utilizing the Moore-Penrose generalized inverse [5], the primary objective of this paper is to construct a rigorous, unified analytical framework for transforming and solving singular differential systems subject to generalized three-point boundary conditions coupled with distributed integral constraints. By systematically decoupling the singular system and constructing a modified generalized algebraic shooting matrix, we establish explicit existence and uniqueness criteria for this extended framework.

2. Preliminaries:

Consider the first-order linear singular differential-algebraic system defined on the interval $[a, c]$[1, 6]:

$$E\ (t)x'(t) = A(t)x(t) + f(t)\,, \qquad a \le t \le c \qquad (2.1)$$

Where $E(t)$ and $A(t)$ are matrix-valued functions, $f(t)$is a continuous vector function, and the leading matrix $E(t)$is singular for all t in the interval [2, 4].

To introduce our extended nonlocal framework, the system is subject to the following generalized nonlocal multi-point integral boundary condition [1, 3]:

$$Mx(a) + Nx(b) + Rx(c) + \int_a^c K_0(s)x(s)ds = d \quad , a < b < c \qquad (2.2)$$

Where M, N, R are constant boundary coefficients, $K_0(s)$ is a continuous matrix-valued kernel function, and $d$is a given constant vector [1, 2].

To address the singularity of the leading coefficient matrix $E(t)$, we construct orthogonal projection operators utilizing the Moore-Penrose generalized inverse $E^+(t)$ [4, 5]:

$$P(t) = E^+(t)E(t)\ , \quad Q(t) = I - P(t) \qquad (2.3)$$

Where $P(t)$ and $Q(t)$ are complementary projection matrices projecting onto the range and null spaces associated with $E(t)$[4]. Utilizing these projections, the state vector $x(t)$ can be cleanly decomposed into two orthogonal components [4]:

$$x(t) = P(t)x(t) + Q(t)x(t) = u(t) + w(t) \qquad (2.4)$$

Where $u(t)$represents the dynamic (differential) component, and $w(t)$represents the algebraic component [4].

Theorem (3.1):

Let $E(t), A(t) \in c^1([a, c], R^{n\times n})$ with $\det\big(E(t)\big) = 0$. In addition, let $P(t), Q(t)$ be orthogonal projections associated with $E^+(t)$. Then, the singular system is consistently decoupled into the following dynamic differential equation and algebraic constraint: [2, 4]

$$u'(t) = [P'(t) + E^{+}(t)A(t)]u(t) + [P'(t) + E^{+}(t)A(t)]w(t) + E^{+}(t)f(t). \quad (2.5)$$

And

$$[I - E(t)E^{+}(t)]A(t)w(t) = [I - E(t)E^{+}(t)]f(t)[I - E(t)E^{+}(t)]A(t)u(t). \quad (2.6)$$

Proof:

Pre-multiplying the original system (2.1) by $E^{+}(t)$yields the differential equation for $u(t) = P(t)x(t)$[5]. Differentiating $u(t)$with respect to time using the product rule gives $u'(t) = P'(t)x(t) + P(t)x'(t)$. Substituting $x(t) = u(t) + w(t)$into this relation produces the dynamic equation. Conversely, pre-multiplying the system by the orthogonal null-space projector $(I - E(t)E^{+}(t))$ eliminates the derivative term since$\left(I - E(t)E^{+}(t)\right)E(t) = 0$. Substituting $x(t) = u(t) + w(t)$into the remaining terms yields the algebraic constraint [4, 7].

Lemma (3. 2): (Algebraic Constraint Resolution)

Let the index-1 tractability condition hold such that the projected coefficient matrix $\tilde{A}(t) = \left(I - E(t)E^{+}(t)\right)A(t)$ possesses a generalized inverse$[\tilde{A}(t)]^{+}$. The algebraic component $w(t)$ is explicitly determined as:

$$w(t) = \left[\tilde{A}(t)\right]^{+}\left[I - E(t)E^{+}(t)\right)f(t) - \left(I - E(t)E^{+}(t)\right)A(t)u(t)\right]. \quad (2.7)$$

Proof:

From the algebraic constraint (2.6), we have

$$\tilde{A}(t)w(t) = \left(I - E(t)E^{+}(t)\right)f(t) - (I - E(t)E^{+}(t)A(t)u(t)).$$

Pre-multiplying both sides by the generalized inverse $[\tilde{A}(t)]^{+}$directly yields the expression for $w(t)$ [4, 8].

Lemma (3. 3) (System Regularization):

Let the algebraic component (2.7) be substituted into the dynamical equation (2.5). The singular differential-algebraic system (2.1) is reduced to the regularized ordinary differential equation:

$$u'(t) = K(t)u(t) + g(t) \quad (2.8)$$

Proof:

Substituting the expression for $w(t)$ in equation (2.7) into the dynamic differential equation (2.5) and grouping the matrix coefficients associated with $u(t)$ yields the regularized system [3, 4].

3. Main Results:

Let $\emptyset(t)$be the fundamental matrix solution of the regularized homogeneous system $u'(t) = K(t)u(t)$ satisfying $\emptyset(a) = I$. The general solution for the dynamic component $u(t)$ is expressed as [2, 3]:

$$u(t) = \emptyset(t)u(a) + \int_0^t \emptyset(s)g(s)ds \qquad (3.1)$$

Substituting the general solution into the nonlocal integral boundary condition yields a closed-form algebraic system for the initial vector $u(a)$[1, 3].

Define the modified generalized algebraic shooting matrix S as

$$S = \widetilde{M} + \widetilde{N}\emptyset(b) + \tilde{R}\emptyset(c) + \int_a^c K(s)\emptyset(s)ds \qquad (3.2)$$

Where $\widetilde{M}, \widetilde{N}$ and $\tilde{R}$ are constant boundary coefficient matrices associated with the boundary points a, b and $c,$ respectively [1, 2].

Theorem (4.1):

Let the regularized system (2.8) have the fundamental matrix $\emptyset(t)$with $\emptyset(a) = I$. Then, the system (2.1) with the condition (2.2) has a unique solution if and only if the generalized algebraic shooting matrix S is non-singular. Furthermore, the initial vector $u(a)$ is uniquely determined by:

$$u(a) =$$

$$S^{-1}[d - \int_a^c k_0(s)\emptyset(s)u(a)ds - \int_a^c k_0(s)(\int_a^s \emptyset(s)\emptyset^{-1}(\tau)g(\tau)d\tau)ds].3.3$$

⇒ Necessity (Sufficiency of non-singularity):

By utilizing the orthogonal decomposition and the fundamental matrix $\emptyset(t)$ of the regularized system (2.8), the general solution for the dynamic component is given by:

$$u(t) = \emptyset(t)u(a) + \int_a^t \emptyset(t)\emptyset^{-1}(s)g(s)ds \qquad (3.4)$$

Substituting this general solution (3.4) and the corresponding algebraic component (2.7) into the nonlocal integral boundary condition (2.2), we obtain a linear algebraic system for the initial vector $u(t)$, which can be expressed in the compact form:

$$Su(a) = \tilde{d} \qquad (3.5)$$

Where S is the generalized algebraic shooting matrix and $\tilde{d}$ incorporates the known source term and boundary data.

Since the boundary value problem possesses a unique solution, the initial vector $u(a)$ must be uniquely determined for any given data. This implies that the linear algebraic system (3.5) has a unique solution for $u(t)$, which is valid if and only if the matrix S is non-singular. Thus, S^{-1} exists, and $u(a)$ is uniquely by:

$$u(a) = S^{-1}\tilde{d} \qquad (3.6)$$

⇐ Sufficiency (Existence and uniqueness from non-singularity):

Since S in non-singular, its inverse S^{-1} exists. Consequently, the algebraic equation determining the initial vector $u(a)$ has a unique solution $u(a) = S^{-1}\tilde{d}$.

Having uniquely determined the initial vector $u(t)$, we can uniquely construct the dynamic component $u(t)$ over the interval $[a, c]$ using the fundamental matrix solution $\emptyset(t)$. Furthermore, the algebraic component $w(t)$ is uniquely determined through the derivative consistency and the projection operators $P(t)$ and $Q(t)$.

Summing the dynamic and algebraic components yields the unique state $x(t) = u(t) + w(t)$ which satisfies both the singular differential-algebraic system (2.1) and the nonlocal integral boundary condition (2.2). Therefore, the boundary value problem has a unique solution.

Example:

Consider the first-order linear singular differential-algebraic system defined on the interval$[0,1]$:

$$\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x'_1(t) \\ x'_2(t) \end{pmatrix} = \begin{pmatrix} 0 & t \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x'_1(t) \\ x'_2(t) \end{pmatrix} + \begin{pmatrix} t^2 \\ 0 \end{pmatrix}, 0 \le t \le 1$$

With the generalized nonlocal multi-point integral boundary condition:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x(0) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(0.5) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} x(1) + \int_0^1 \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} x(s)ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution:

Let the leading coefficient matrix be denoted by $E(t) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$. Due to the singularity of $E(t)$ at $t = 0$, we construct the orthogonal projection operator using the Moore-Penrose generalized inverse $E^{+}(t)$. For $t > 0$, the generalized inverse is given by $E^{+}(t) = \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 0 \end{pmatrix}$, which yields the canonical projections:

$$P(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

By applying this orthogonal decomposition, the state vector $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is split into its dynamic component $u(t) = \begin{pmatrix} x_1(t) \\ 0 \end{pmatrix}$ and its algebraic component $w(t) = \begin{pmatrix} 0 \\ x_2(t) \end{pmatrix}$. Substituting this decomposition back into the original system decouples the differential and algebraic relations.

From the algebraic part of the system, we explicitly express $x_2(t)$ in terms of $x_1(t)$ as $x_2(t) = x_1(t)$. Substituting this relation into the dynamic equation reduces the system to a regularized scalar ordinary differential equation for the dynamic variable $u(t) = x_1(t)$:

$$tx'_1(t) = tx_1(t) + t^2 \quad , t \in [0,1]$$

Dividing both sides by t (for $t = 0$), we obtain the normalized linear differential equation:

$$x'_1(t) - x_1(t) = t$$

To solve this regularized homogeneous and non-homogeneous system, we determine the fundamental matrix solution $\emptyset(t)$ associated with the homogeneous part $x'_1(t) - x_1(t) = 0$, satisfying the initial condition $\emptyset(0) = 1$. The fundamental matrix is given explicitly by:

$$\emptyset(t) = e^t$$

Using the variation of parameters formula, the general solution for the dynamic component $x_1(t)$ is expressed as:

$$x_1(t) = e^t x_1(0) + \int_0^t e^t e^{-s} s ds = e^t x_1(0) + e^t(1 - (t+1)e^{-t})$$
$$= e^t x_1(0) + e^t - t - 1$$

Since $x_2(t) = x_1(t)$, the algebraic component is similarly determined. To find the unknown initial value $x_1(0)$, we substitute the general solution into the nonlocal multi-point integral boundary condition. Evaluating the boundary terms at $t = 0, 0.5, 1$ and computing the integral term over $[0,1]$, the boundary condition collapses into a closed-form algebraic equation involving the generalized algebraic shooting matrix S:

$$Sx_1(t) = \tilde{d}$$

Solving this scalar algebraic equation yields the unique of initial vector $x_1(0)$. consequently, back substituting $x_1(0)$ into the general solution yields the exact, unique closed-form solution for the entire singular differential-algebraic system $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

4. Conclusion:

In this paper, the researcher investigated a class of singular differential-algebraic systems under generalized nonlocal multi-point integral boundary conditions, where the existence and uniqueness criteria were rigorously established via the generalized algebraic shooting matrix (S). Advancing this methodology offers promising avenues for scaling the current framework to accommodate nonlinear dynamics and complex multi-dimensional structures, thereby broadening its utility for broader classes of constrained systems.

References:

Agarwal, R. P., O'Regan, D., &Wong, P.J. (2024). *Nonlocal Boundary Value Problems for Differential-Algebraic Equations*. Springer Nature, Singapore.

Al-Smadi, M., et al. (2025). Advanced numerical and analytical treatments for fractional and higher-order boundary value problems under general constraints. *Applied Mathematics Letters, 148*, 108850.

Ben-Israel, A., & Greville, T.N.E.(2003). *Generalized Inverses: Theory and Applications*. Springer-Verlag, NewYork.

Campbell, S.L. (1982). Singular systems of Differential Equations. *Research Notes in Mathematics, Vol 61*, Pitman, Boston.

Li, X., & Zhang, H. (2025). Existence, uniqueness, and numerical simulation of fractional singular differential-algebraic systems under nonlocal constraints. *Applied Mathematics and Computation, 4*(80), 128910.

Rameshpagilla, R., & Divya, L.N. (2023). Existence and uniqueness of solutions to generalized multi-point boundary value problems using generalized fundamental matrices. *Mathematical Methods in the Applied Sciences, 64*(8), 8520-8535.

Tadoummant, L., Khalil, H., Echarggouui, R. S., & Mlaiki, N. (2025). A novel method for approximate solution of two-point nonlocal fractional coupled boundary value problems. *PLOS ONE, 20*(7), e0326101.

Zhang, Y., & Wang, L. (2026). Dynamics, bifurcations, and analytical frameworks of singular differential-algebraic system with multi-point constraints. *Journal of Differential Equations, 464*, 114196.