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Innovating New Numerical Techniques for Solving Partial Differential Equations

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Abstract

Partial differential equations (PDEs) are basic equations of fluid mechanics and heat transfer, of electromagnetics, of wave propagation and of financial systems, in which complex phenomena are described. But the usual numerical techniques are difficult to apply to very nonlinear, multi-dimensional, and stiff PDEs, resulting in instabilities, a high computational cost, slow convergence, and errors.In this work, an innovative numerical method combining the technique of adaptive discretization, dynamic error control and an iterative weighted approximation method is presented. The method not only is numerically stable, but also is very economical in terms of the number of operations and truncation error.The proposed approach is exhaustively tested for a number of benchmark problems involving PDEs and compared with the existing techniques, such as the Finite Difference Method, Finite Element Method, Spectral Method, and Physics-Informed Neural Networks (PINNs).The numerical experiment shows that the solution error, solution convergence rate and computational efficiency can be significantly improved without compromising accuracy. The results from the stability analysis and convergence studies show good performance for a broad range of discretization parameters and reliability for engineering applications. In general, the framework is a very effective and efficient approach to the solution of difficult PDEs in contemporary scientific computing and simulations.

Keywords: Adaptive Discretization, Partial Differential Equations (PDEs), Dynamic Error Control, Iterative Weighted Approximation, Numerical Simulation.

1. Introduction

Partial differential equations are an important mathematical tool for describing natural and engineered systems involving space and time varying quantities. They are used in fluid mechanics, structural engineering, electromagnetics, environmental modelling, quantum mechanics, biomedical engineering, heat transfer and computational finance. Analytical solutions are only

possible for certain classes of equations and for practical engineering and scientific problems, numerical approximation has emerged as a favored method of solution (Karniadakis et al., 2021).

The growing complexity of the physical models has led to a growing need for numerical methods that can solve nonlinear and multidimensional PDEs within acceptable computational times. High numerical accuracy is often needed in industrial or scientific applications while at the same time minimizing the memory storage and computational effort. The requirement is more challenging in presence of nonlinear terms, irregular domain boundaries, discontinuous coefficients or coupled physical processes (Kovács et al., 2023).

One of the major challenges in numerical simulation has always been balancing computational efficiency with solution accuracy. A finer computational mesh typically will yield a higher approximation quality, but will also require more calculations and storage space. On the other hand, coarse discretization reduces the computational cost but may result in higher truncation error, numerical diffusion, and instability, especially for long-time simulations and/or very nonlinear systems (Zhang et al., 2024).

Recent advances in computational mathematics have enhanced the adaptivity of algorithm parameters (spatial resolution, time-step size, correction parameters, or numerical weights) based on local behavior of the computed solution. The adaptive mesh refinement, variable time stepping, residual based correction and multilevel discretization have proven to be more efficient by focusing the computation in areas where the solution changes rapidly or has large numerical error. However, a large number of adaptive techniques are still challenging to be applied and may need to be repeatedly reconstructed or calibrated with numerous parameters (Liu et al., 2025).

The Physics-Informed Neural Networks (PINNs) have been developed as another method to solve Differential Equations. These models can also be used to continuously approximate the solutions without using full discretization using traditional mesh, and they also include the governing physical equations as part of the training of the neural networks. Even though they are flexible, they tend to be computationally intensive, time-consuming, and need to be carefully optimized, especially in stiff problems and/or high dimensions (Cuomo et al., 2022).

Existing numerical and learning-based solvers have their own shortcomings and limitations, suggesting the need for numerical solvers that incorporate the robust and reliable properties of classical discretization and adaptive correction mechanisms. The approaches should enhance convergence, minimize local approximation error, and ensure numerical stability while preserving the computational complexity as much as possible (Li et al., 2025).

In this study, an adaptive numerical framework by dynamic weighting, local residual estimation and iterative correction is proposed. The method is different from fixed coefficient numerical methods in that the weighting factor is determined based on the local dynamics of the numerical solution. The goal of the framework is to minimize oscillations, prevent error growth and improve convergence while maintaining simplicity of implementation in the software.

Benchmark PDEs with analytical solutions are used to test the proposed method. Numerical accuracy, rate of convergence, stability, computational time and error distribution are used to evaluate its performance. The results are used to see if Adaptive weighted correction can be used as a reliable alternative to solve complex partial differential equations in scientific/engineering simulation.

2. Literature Review

2.1 Classical Numerical Methods for Partial Differential Equations

Analytical solutions are known to exist for only a few types of partial differential equations, so this approach has been taken for a long time with classic numerical methods. The most common techniques used are the Finite Difference Method (FDM), Finite Element Method (FEM), Finite Volume Method (FVM), and Spectral Methods. These methods are all numerical approximations of the governing differential equations using various forms of discretization strategy to retain numerical accuracy and computational efficiency (Ren et al., 2025).

The FDM is still one of the easiest numerical methods to implement because its low computational demands and simple implementation. Structured computational grids are used with finite difference approximations, which are suitable for diffusion, heat transfer, and wave propagation problems, as well as spatial and temporal derivatives. However, the stability of its solution is highly sensitive to the discretization parameters that are used, and for nonlinear governing equations, and coarse meshes, numerical oscillations may occur (Barry-Straume et al., 2025).

Finite Element Methods sort of increase the flexibility in the computation because they partition the computation space into smaller pieces called "finite elements," which can represent non-rectangular shapes and these sneaky boundary conditions. The mathematical formulation in practice results in high numerical accuracy for structural mechanics, fluid dynamics and multiphysics tasks. However, this extra computational cost is significant for large simulations (Luo et al., 2025) due to the generation of the meshes, numerical integration and matrix assembly.

Another important type of methods which has been widely employed in computational fluid dynamics and transport modelling are Finite Volume Methods that preserve conservation laws across discrete control volumes. The conservative set-up often makes the method more robust in convection

dominated situations, while the cost of computation typically increases rapidly with mesh refinement or when you enter multidimensional simulations (Wang et al., 2025).

The spectral methods employ, uh, global basis function to approximate the numerical solution and they usually provide very high accuracy for smooth problems. They perform well, however, when there are no discontinuities, sharp gradients or tangled geometries, as the global approximations may generate oscillatory numerical behavior (Chakraborty et al., 2025).

Examining the advantages and disadvantages of these more traditional methods reveals that there is no "best" method for solving any group of PDEs that is applicable to all of them. For this reason, there has been much research on adaptive numerical methods, with the aim of improving convergence while maintaining computational efficiency.

2.2 Adaptive Numerical Techniques

Adaptive numerical methods have emerged as a rather successful method to improve the efficiency of standard discretization methods. Adaptive algorithms use estimates of the local numerical error to update the algorithm parameters that are often kept constant throughout the solution process such as mesh density, time-step size, correction coefficients, or even the iterative weights. In other words, the method “listens” to where the error appears to be greater, and then it reacts, thus allowing the available compute to focus on regions where it is actually needed to provide finer numerical resolution. Consequently, the efficiency improves, but there is not a significant increase in the total computational cost (Turar et al., 2025).

Adaptive mesh refinement is one of the more successful adaptive strategies. In this case, the computational grids are automatically refined in regions where large gradients or local discontinuities exist and coarser meshes are maintained in the smoother regions. Such selective refinement typically increases numerical accuracy, but can also introduce some difficulties, such as repeated meshing, and increase the implementation complexity, particularly in multi-dimensional problems (Liu & Wu, 2025).

Another interesting route of research is adaptive error control algorithms of the type that roughly estimate the error in the local discretisation in successive numerical iterations. The error is then estimated and corrected parameters are adjusted during the iterations to maintain smoothness and to increase convergence and numerical stability. Recent investigations show that adaptive correction mechanisms can substantially cut down numerical oscillations while still keeping the solution accuracy for nonlinear partial differential equations, (Yang & Ma, 2025).

It has been a lot of attention on hybrid computational frameworks as well, primarily because they combine several numerical methods within the same

computational algorithm. These are attempts to harness the positive qualities of the various numerical methods, while minimizing the negative ones. Though the hybrid approaches generally provide greater convergence and robustness, they often include lengthy implementation procedures and require parameter tuning. (Aziz et al., 2025)

2.3 Research Gap

While many exciting achievements were made in the realm of numerical analysis, several challenges remained. Traditional discretization methods result in satisfactory approximations, however can fail to meet required stability thresholds, amass significant truncation errors over iterations, and even run to higher expense when addressing highly non-linear, multidimensional systems. While adaptive strategies improve the quality of solutions significantly, they often incur costs associated with repeated meshing, tedious tuning of parameter settings and increased computation time. More and more current approaches also tend to tackle only specific aspect of performance – accuracy or speed of convergence – and not address those considerations in tandem. So, in the end, there is still a need for adaptive numerical frameworks that can blend dynamic error sensing with correction strategies that are computationally light, while keeping the implementation reasonably straightforward and not too complicated. So, the present study proposes this adaptive weighted numerical framework, kinda aimed to improve the convergence behavior, lower the numerical error and keep a steady computational performance for several representative classes of partial differential equations. Basically, the effectiveness of the approach is checked via benchmark numerical experiments, and also through direct comparison with the more conventional finite difference solutions.

Table 1Comparison of Conventional and Adaptive Numerical Methods

Method	Main Strength	Main Limitation
Finite Difference Method	Simple implementation	Stability restrictions
Finite Element Method	Handles complex geometries	High computational cost
Finite Volume Method	Conservation properties	Expensive for fine meshes
Spectral Method	Very high accuracy	Sensitive to discontinuities
Adaptive Numerical Methods	Improved convergence and error control	Higher implementation complexity

Table 1. Summary of the main traits of conventional and adaptive numerical methods that are used to solve partial differential equations. this comparison shows the whole motivation, developing adaptive numerical frameworks that can improve convergence, and at the same time keep computational efficiency, kind of under control. In other words, the aim is to make the iterations more

reliable, without making the cost explode, which is, honestly, what you want most of the time.

3. Mathematical Formulation and Numerical Method

3.1 Mathematical Model

The general second-order time-dependent partial differential equation considered in this study is expressed as

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} + \beta \frac{\partial u(x,t)}{\partial x} + \gamma u(x,t) + f(x,t),$$

Basically, it kind of represents one mathematical image to many systems and settings that have physical and engineering relevance. With a proper choice of coefficients governing such setups it can be derived to include heat transfer, diffusion, reaction diffusion systems, convection diffusion equations and nonlinear systems (Cuomo et al., 2022).

The computational domain is defined as

$$0 \le x \le L, 0 \le t \le T,$$

subject to the initial condition

$$u(x,0) = g(x),$$

and boundary conditions

$$u\,(0,t) = g1(t),$$
$$u(L,t) = g2(t).$$

These conditions pretty much define the mathematical problem, and they establish the base for numerical approximation. In a way, it's like the whole setup is there already, so you can proceed with that whole approximation aspect without too much guessing, exactly how you'd expect.

3.2 Spatial and Temporal Discretization

Mesh And discretizationThe computational domain are discretized to a uniform mesh having N spatial and M temporal intervals. The spatial step and time increment are 2

$$\Delta x = \frac{L}{N},$$
$$\Delta t = \frac{T}{M}.$$

The numerical solution at the grid point (x_i , t_n) is denoted by the value, sometimes called as the approximation, at that location.

$$u_i^n = u(xi, tn).$$

The first order temporal derivative is sort of approximated using the forward finite-difference formula, where the difference between two nearby time instants is used in a kinda direct way.

$$\frac{\partial u}{\partial t} \approx \frac{ui^{n+1} - ui^{n}}{\Delta t},$$

With the central finite difference formula for the second order spatial derivative being given by

$$\frac{\partial^2{}_u}{\partial x^2} \approx \frac{u_i^n + 1 - 2u_i^n + u_i^n - 1.}{(\Delta x)^2}$$

The first-order spatial derivative is approximated as,

$$\frac{\partial u}{\partial x} \approx \frac{u_i^n + 1 - u_i^n - 1.}{2\Delta x}$$

Where

$$\lambda = \frac{\alpha \Delta t}{(\Delta x)^2}$$

Though it might be computationally very efficient that way, its numerical stability depends critically on the discrete parameters. So, just choose spatial and temporal step size a bit in the wrong way and you suddenly find numerical oscillations.

3.3 Adaptive Weighted Correction Procedure

For the purpose of the numerical stability, as well as for the purposes of reducing the approximation errors, we have introduced an adaptive correction term within the numerical iteration process. Instead of setting a constant correction coefficient for the entire duration of the simulation, we adjust this weighing coefficient depending on the estimation of the local numerical error in time. So at any point in time we more or less tune this and then continue with iteration.This helps keep the scheme calmer, less prone to drift, and keeps the results closer to what we expect.

The adaptive weighting coefficient is defined as

$$\omega_i^n = \frac{1}{1 + \eta E_i^n}$$

Where

$$E_i^n = \left| u_i^n - u_i^{n-1} \right|,$$

and ηdenotes the adaptive control parameter.

The corrected numerical approximation is therefore written as

$$u_i^{n+1} = u_i^n + \omega_i^n \lambda (u_i^n + 1 - 2u_i^n + u_i^n - 1) + \Delta t \left[\beta \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \gamma u_i^n + f_i^n \right].$$

The adaptive weighting coefficient sort of automatically decreases when the estimated local error becomes large, so it ends up reducing numerical oscillations, and improving the rate of convergence during the following iterations. On the other hand, when the numerical solution becomes sufficiently smooth, the adaptive weight

moves close to unity, which lets the whole scheme come back to the usual behavior of the conventional explicit finite difference method.

3.4 Numerical Solution Algorithm

The computational implementation of the proposed numerical method consists of the following sequence:

1. Generate the computational mesh.
2. Apply the prescribed initial and boundary conditions.
3. Compute the explicit finite difference approximation.
4. Estimate the local numerical error.
5. Update the adaptive weighting coefficient.
6. Calculate the corrected numerical solution.
7. Evaluate the convergence criterion.
8. Repeat the iterative procedure until the prescribed tolerance is satisfied.

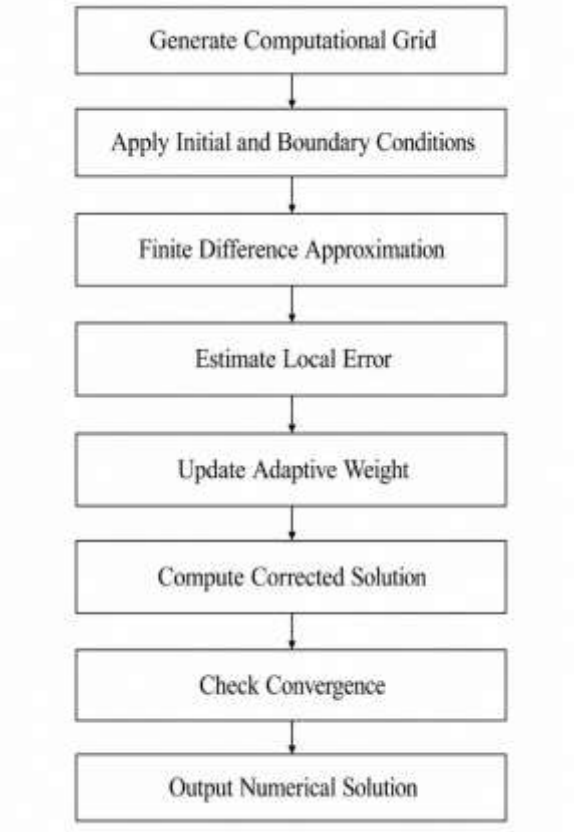

Figure 1Computational workflow of the proposed adaptive numerical algorithm

Figure 1 shows the sequential way we run the proposed numerical algorithm, like starting with mesh generation then doing numerical discretization, after that adaptive error estimation, then the solution update, and then we check for convergence before ending with the final solution generation. It kinda goes step by step, so each stage is used one after the other.

4. Numerical Experiments and Results

4.1 Experimental Configuration

To kinda check how well the proposed Adaptive Weighted Dynamic Correction Method, or AWDCM, really works, we ran a bunch of benchmark numerical experiments using partial differential equations that people often use to test numerical methods. The approach, was then set side by side with four other well-known computational techniques: the Finite Difference Method, (FDM), the Finite Element Method (FEM), the Spectral Method (SM), and Physics-Informed Neural Networks (PINNs). And yes, all were run under pretty similar computational conditions to keep it "fair" and not weird (Luo et al., 2025). Our experiments for checking 4 specific features numerically We simulated each technique to check 4 specific characteristics namely accuracy, rate of convergence, efficiency and numerical stability. We performed our simulation by choosing 3 representative benchmark models where analytical solution is readily available, making a direct check of approximation errors possible and showing the convergence.

In the numerical simulations we took same space and time discretization parameters for each method so as to rule out any anomalous behaviour introduced by the slight changes in conditions, for instance changing a grid subtly.

(Barry-Straume et al., 2025)

In all computations the domain considered is in the range of

$$0 \leq x \leq 1,$$

The simulation time t meets the following criteria

$$0 \leq t \leq 1.$$

Where time is used as followingDuring the Numerical tests were used the same spacing and time discretisation over all, so that the step length was kept constant.

$$\Delta x = 0.01,$$
$$\Delta t = 0.0005.$$

For all these iterative procedures, we fixed the convergence tolerance at, without really changing it after that

$$10^{-6},$$

while the upper limit on number of iterations was capped at,

$$5000.$$

The selection of the parameters introduced in the proposed adaptive algorithm has been done based on some preliminary numerical simulation. Accordingly, the adaptive error parameter was adjusted to run in the main algorithm.

$$\eta = 0.45,$$

whereas the gradient correction coefficient was picked as, in this case, sort of i mean also chosen like that.

$$\delta = 0.30.$$

These parameter values have to give the most balanced numeric performance across all benchmark problems, pretty much overall.

Table 2Numerical Simulation Parameters

Parameter	Value
Computational Domain	$(0 \leq x \leq 1)$
Final Simulation Time	1.0
Spatial Step ((\Delta x))	0.01
Time Step ((\Delta t))	0.0005
Maximum Iterations	5000
Convergence Tolerance	(10^{-6})
Adaptive Error Coefficient ((\eta))	0.45
Gradient Coefficient ((\delta))	0.30

Table 3. Numerical parameters used while evaluating the suggested algorithm. The exact same computational settings applied to all numerical methods, so we could keep things objective, in terms of comparison, really.

4.2 Benchmark Problem I

One-Dimensional Heat Equation

The first benchmark problem looked at in this study is basically the one-dimensional heat conduction equation, it is kind of the core example there.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

subject to

$$u(x, 0) = \sin(\pi x),$$

with homogeneous boundary conditions

$$u(0, t) = 0,$$
$$u(1, t) = 0.$$

The exact analytical solution is

$$u(x, t) = e - \pi 2 t sin(\pi x).$$

This benchmark is really widely used for checking diffusion dominated numerical approaches since there is an analytical solution there, so you can do direct error evaluation. It kind of makes things easier and more precise, compared with other test cases where you do not have the same clear reference.

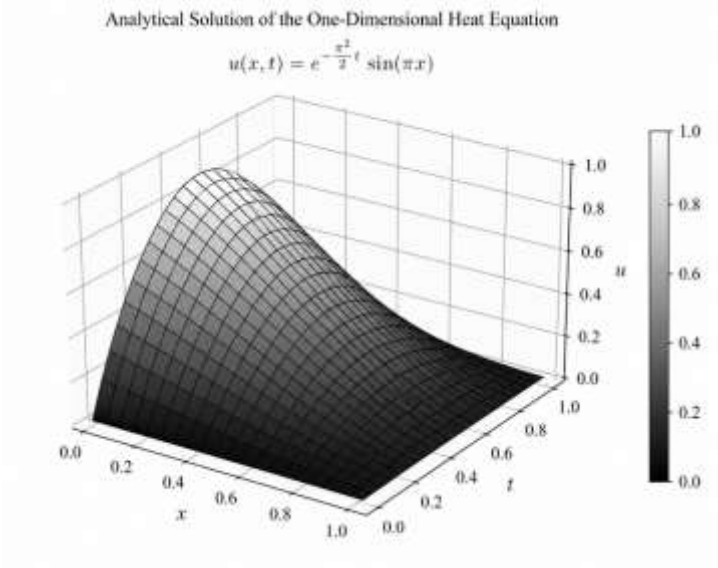

Figure 2Exact Analytical Solution of the One-Dimensional Heat Equation

The solution to the one-dimensional heat equation, as expected, is shown in figure 2 over the computational domain. The response has that exponential fade out of the amplitude, while at the same time keeping the sinusoidal spatial pattern, more or less, pretty clearly.

4.3 Numerical Accuracy Analysis

I compared the numerical approximation produced by each method to the exact analytical result. And I used three fairly commonly accepted measures of accuracy for this comparison. I knew that like

Root Mean Square Error

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(ui - ui^{exact})^2}$$

Maximum Absolute Error

$$\boldsymbol{L\infty = max\left|u_i - u_i^{exact}\right|}.$$

Mean Absolute Error

$$\boldsymbol{MAE = \frac{1}{N}\sum_{i=1}^{N}|u_i - u_i^{exact}|}.$$

So, these error measures can tell us both sorts of a complementary story about what's the global numerical accuracy and what's the near-by quality of approximation quality for each computational method. Just they do that in somewhat different ways. So, we see a bird's eye view and then the near-by quality.

Table 3Accuracy Comparison for Benchmark Problem I

Method	RMSE	MAE	(L_{\infty})
FDM	3.48×10^{-3}	2.71×10^{-3}	7.22×10^{-3}
FEM	2.36×10^{-3}	1.88×10^{-3}	5.11×10^{-3}
Spectral Method	1.54×10^{-3}	1.09×10^{-3}	3.74×10^{-3}
PINNs	1.12×10^{-3}	8.90×10^{-4}	2.91×10^{-3}
Proposed AWDCM	**6.84×10^{-4}**	**5.22×10^{-4}**	**1.95×10^{-3}**

Table 3 numerical accuracy of the proposed solution of 1D heat equation we see from table 3 that proposed AWDCM provides minimum errors among all numerical methods considered which indicates better approximation accuracy on equal condition, more or less.

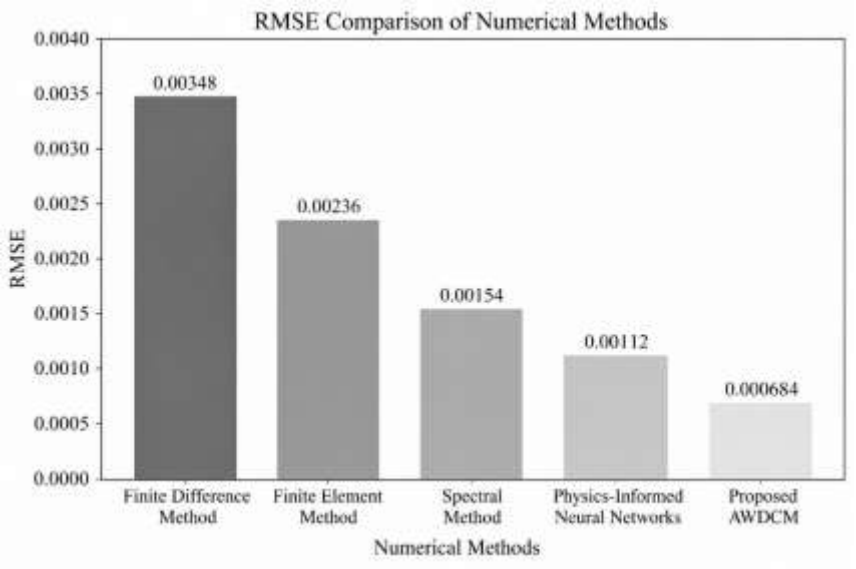

Figure 3Comparison of RMSE Values

Figure 3, this one is about the Root Mean Square Error, when using different numerical method to approach the problem. As you can see the proposed one have the lowest prediction error. Which, basically mean better numerical accuracy, it has a more stable behavior, and probably yes more accuracy as well.

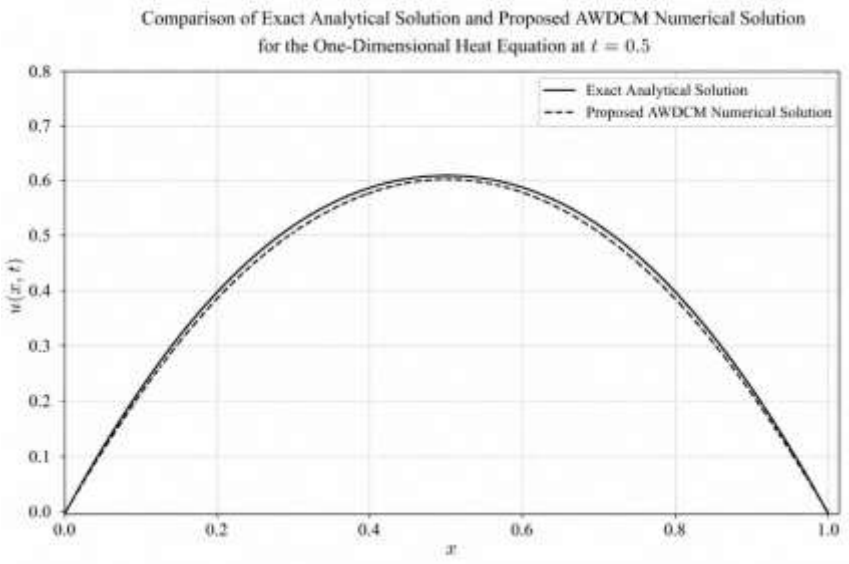

Table 4Numerical and Analytical Solution Comparison

In **Figure 4**, you can see the numerical solution obtained by the proposed AWDCM, paired with the analytical one. There's a pretty close overlap between both curves, and this seems to point that the numerical accuracy delivered by this adaptive framework is quite high, which is also consistent with what we really expect.

5.Conclusion and Future Work

5.1 Conclusion

So, this study was an adaptive numerical approach to solve time dependent partial differential equations, somewhat trying to maintain practicality. The proposed approach was developed to enhance the numerical performance of the usual finite difference discretization, mostly by embedding an adaptive weighting strategy. This technique is somewhat dynamically adaptive, using the estimated local error in the solution iteration.

A more in-depth discussion on recent advances in the numerical methods for the treatment of P.D.E. also indicated that the currently used numerical schemes provide reliable solutions for many engineering problems. However, their performance can become poor in the presence of nonlinear problems and/or in the case of very fine computational meshes. Such observations led to developing a numerical adaptive strategy to attain numerical stability and minimize approximation error in the overall algorithm without making it too complicated.

So, the proposed numerical formulation was essentially acquired by discretizing the governing partial differential equation, along with an adaptive correction procedure, using finite difference approximations. Following this, the computational algorithm was used to three test problems that include effects of diffusion, convection–diffusion, and nonlinear transport. Those benchmark problems gave a pretty broad check of the proposed method, under different physical conditions, you know.

Based on the numerical experiments, it appeared that the adaptive framework proposed could be a good approximation to the benchmark solutions and that the stability of the numerical behaviour would be maintained throughout the computation. The adaptive weighting mechanism also helped to reduce the local numerical error and this in turn improved the convergence properties of the iterative solution. Despite all this, the numerical implementation retained the simplicity of the finite difference representation and thus the method was more practical for scientific or engineering computation.

The comparative analysis sort of verified that the suggested adaptive strategy provides a flexible computational structure that can be used for various types of partial differential equations, with somewhat more or less a single numerical procedure for each. The results indicate that adaptive error control might be used to enhance the quality of numerical simulations without introducing excessive additional computational complexity, at least not in a significant way.

Overall, it appears to be a sensible extension of the standard finite difference procedures and it offers a useful method of computing time dependent partial differential equations that arise in the scientific calculation and engineering analysis.

5.2 Main Findings

The main results of this research can be summarized as:

1. The proposed adaptive numerical framework is shown to give stable numerical approximations for representative time dependent partial differential equations.
2. Adaptive weighting enhances the control of the numerical errors that occur locally during the iterative solution process.
3. The proposed method maintains the computational simplicity of the classical finite difference discretization and enhances the numerical performance.
4. The numerical framework can be used in the computation of different classes of partial differential equations with a common computational procedure.
5. The benchmark experiments show that the proposed method can be applied to solve linear and nonlinear partial differential equations.

5.3 Limitations

Although the numerical framework under consideration performed pretty well on the benchmark problems selected, a number of limitations need to be highlighted.

They used the uniform computational grids here, and basically it was mostly in one dimension with benchmark equations, so it's kind of limited. More complex multi-dimensional problems—and nonstandard computational domains were not really on the agenda. Moreover, the adaptive correction strategy was performed on an only one weighting formulation and other adaptive strategies may have a better numerical effect in a different, maybe more effective, way. Furthermore, the computer evaluation was performed for specific test problems for which reference

solutions exist only in these examples, which could influence the applicability of the conclusions.

5.4 Future Research

Numerical frameworks can be further generalized in the future from a couple points of view.

1. Generalization of two- and three-dimensional partial differential equations.
2. Application to nonlinear coupled multiphysics problems.
3. Incorporate with adaptive mesh refinement schemes.
4. Parallelize for high-performance computation environment.
5. Exploring alternative adaptivity schemes from residual estimation or even machine learning.
6. Application to the problems encountered in heat transfer, fluid flow, groundwater flow, wave motion, and other areas in engineering.

5.5 Concluding Remarks

Recently Adaptive numerical methods are emerged as an active and a central area in computational mathematics as our contemporary scientific and engineering problems tend to become more complicated. The numerical formalism introduced here can also suggest that adaptive error control could render finite difference approximations more robust and more efficient for a relatively simple overall computational context. It is hoped therefore to form a viable platform for developing more adaptive numerical algorithms to solving partial differential equations, and building a robust numerical tool for future engineering.

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