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Adaptive Spectral Homotopy Perturbation Method for Long-Time Integration of Fractional Partial Differential Equations: Rigorous Treatment of the Memory Effect

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Abstract

Fractional partial differential equations (FPDEs) are widely used to model systems with memory effects and anomalous diffusion, yet their analytical solutions are rarely available. This paper presents an Adaptive Spectral Homotopy Perturbation Method (A-SHPM) for the accurate and stable long-time solution of FPDEs. The proposed method extends the Spectral Homotopy Perturbation Method (SHPM) through systematic time-domain decomposition while rigorously preserving the memory effect of the Caputo fractional derivative across successive sub-intervals. The history term is evaluated numerically using Gauss–Lobatto quadrature based on approximate solutions from previous sub-intervals, eliminating the need for the exact solution. A convergence analysis based on the Banach Fixed Point Theorem proves that the history term preserves the contraction property of the iterative scheme, and explicit criteria for selecting sub-interval sizes are derived. Numerical results for time-fractional diffusion and reaction–diffusion equations demonstrate that A-SHPM maintains high accuracy and numerical stability for long-time integration up to (t=5.0), significantly extending the applicability of SHPM beyond its theoretical stability limit. At an error tolerance of (10^{-4}), the proposed method achieves accuracy comparable to finite difference methods with substantially lower computational cost.

Keywords: : Fractional Partial Differential Equations; Adaptive Spectral Homotopy Perturbation Method; Caputo Derivative; Long-Time Integration; Memory Effect; Convergence Analysis..

1. Introduction

The field of fractional calculus, encompassing integral and differential operations of non-integer order, traces its origins to the correspondence between Leibniz and L'Hôpital in 1695, where the concept of a half-derivative was first mentioned. For centuries, fractional calculus remained largely within the domain of pure mathematics, viewed as a theoretical generalization without immediate physical application. However, over the last few decades, this perspective has shifted dramatically as researchers recognized that fractional calculus provides an efficient and excellent instrument for describing many dynamical phenomena arising in scientific disciplines such as physics, chemistry, biology, economics, viscoelasticity, electrochemistry, electromagnetics, control theory, and porous media transport [11, 12, 15].

Traditional integer-order calculus assumes that the rate of change of a system depends only on its current state. However, many complex systems in nature exhibit memory and hereditary properties, where the future state of the system depends not only on the present state but also on its historical states. Fractional derivatives, being non-local operators, inherently capture these memory effects, making them superior to integer-order derivatives for modeling such systems [11, 12]. This capability has led to the widespread adoption of fractional calculus in fields where anomalous diffusion, viscoelastic behavior, and long-range dependencies dominate the system dynamics.

Among the various mathematical structures in fractional calculus, fractional partial differential equations (FPDEs) appear with increasing frequency in various areas and engineering applications [11, 12]. These equations involve partial derivatives of fractional order with respect to time, space, or both, and are used to model complex spatiotemporal dynamics where both memory effects and non-local spatial interactions are present. Applications span anomalous diffusion in porous media, viscoelastic material behavior, biological tissue mechanics, and financial market dynamics with long-range correlations.

Despite their modeling power, fractional partial differential equations present significant computational challenges. Most such equations lack closed-form analytical solutions, necessitating the development of effective numerical and semi-analytical methods. Over the past two decades, numerous approaches have been proposed, including finite difference methods [2], finite element methods, spectral methods, and various semi-analytical techniques such as the Adomian Decomposition Method (ADM), the Homotopy Analysis Method (HAM), and the Homotopy Perturbation Method (HPM) [4, 5].

The Spectral Homotopy Perturbation Method (SHPM) was recently introduced [1] as a hybrid approach combining the simplicity of the Homotopy Perturbation Method with the exponential convergence properties of Chebyshev pseudo-spectral transformations. The method has demonstrated exceptional accuracy for linear fractional problems, achieving machine precision ($\sim 10^{-15}$) in applications to diffusion equations and fluid dynamics. Theoretical analysis established convergence conditions based on the Banach Fixed Point Theorem, providing explicit error bounds for the truncated series solution.

However, practical application of the standard SHPM revealed a significant limitation: numerical instability for long-time integration. The convergence condition derived in the theoretical analysis requires the time interval T to satisfy $T < T\_max$, which for typical problems with $\alpha = 0.8$ restricts stable computation to $t \leq 0.053$. Beyond this threshold, the alternating terms in the homotopy series grow large before decaying, leading to catastrophic cancellation when summed using finite-precision arithmetic.

This limitation poses a substantial barrier to practical applications. Many physical phenomena modeled by fractional equations evolve over extended time scales. Anomalous diffusion in porous media may require simulation over hours or days. Viscoelastic creep experiments span comparable durations. Financial option pricing requires evaluation at maturity times of months or years. The standard SHPM, restricted to short time intervals, cannot address these applications without modification.

Several researchers have addressed similar convergence limitations in homotopy-based methods through various strategies. The Homotopy Analysis Method introduced an auxiliary convergence-control parameter, though selecting its optimal value requires additional computational effort. Multi-step homotopy approaches divide the time domain into sub-intervals, applying the homotopy method sequentially across intervals. Domain decomposition methods partition both spatial and temporal domains, solving local problems that are then matched at interfaces. Recent work by Wang et al. [18] and Chen et al. [19] explored time-domain decomposition methods for fractional evolution equations, but the rigorous treatment of the memory effect in the context of spectral homotopy methods remains an open challenge.

This paper proposes an Adaptive Spectral Homotopy Perturbation Method (A-SHPM) that extends SHPM to arbitrary time domains through systematic time-domain decomposition while rigorously preserving the memory effect. The approach divides the temporal domain into sub-intervals where the convergence condition is satisfied, using the solution at each sub-interval endpoint as the initial condition for the subsequent interval. Unlike standard time-marching schemes, A-SHPM explicitly incorporates the Caputo history term, computed numerically via Gauss-Lobatto quadrature from previously stored approximate solutions, maintaining the semi-analytical character of SHPM within each sub-interval, preserving the method's accuracy advantages while extending its temporal applicability.

The primary contributions of this work are: (1) formulation of the adaptive time-domain decomposition strategy for SHPM with explicit treatment of the memory effect; (2) numerical computation of the history term via Gauss-Lobatto quadrature using approximate solutions from previous sub-intervals, making the method fully predictive; (3) rigorous convergence analysis in a well-defined Banach space, accounting for error propagation across sub-intervals and proving that the history term preserves the contraction mapping property; (4) explicit criteria for determining optimal sub-interval sizes based on problem parameters; and (5) numerical validation through applications to fractional diffusion and reaction-diffusion equations demonstrating stable computation for $t \leq 5.0$.

The remainder of this paper is organized as follows. Section 2 reviews mathematical preliminaries of fractional calculus. Section 3 reviews the standard SHPM formulation and analyzes the source of long-time instability. Section 4 presents the adaptive A-SHPM formulation with detailed algorithmic steps and pseudocode. Section 5 provides rigorous convergence analysis for the adaptive scheme in a Banach space framework. Section 6 presents numerical applications and comparative results at unified error tolerance. Section 7 concludes with summary and future directions.

2. Mathematical Preliminaries

2.1 Fractional Calculus Definitions

Definition 2.1 (Riemann-Liouville Fractional Integral). The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$of a function $f(t)$is defined as [11]:

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \; \alpha > 0, I^0 f(t) = f(t)$$

where $\Gamma(\cdot)$denotes the Gamma function defined by:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt, \; \mathrm{Re}(z) > 0$$

Definition 2.2 (Caputo Fractional Derivative). The Caputo fractional derivative of order $\alpha$is defined as [11, 12]:

$$D^{\alpha}f(t) = I^{m-\alpha}\frac{d^m}{dt^m}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau$$

where $m-1 < \alpha \leq m, m \in \mathbb{N}$.

Remark 2.3. The Caputo derivative satisfies the property that the derivative of a constant is zero, which aligns with classical calculus, unlike the Riemann-Liouville derivative. This property is crucial when modeling physical systems where a constant state should imply zero rate of change [11, 12].

Property 2.4 (Non-locality). The most important property of the Caputo derivative for this work is its non-local nature. The derivative at time $t$depends on the entire history of the function from 0to t. This property is the source of both the modeling power of fractional equations and the computational challenges addressed in this paper.

Property 2.5 (Linearity). The Caputo fractional derivative satisfies the linearity property [11]:

$$D^{\alpha}\big(c_1 f(t) + c_2 g(t)\big) = c_1 D^{\alpha} f(t) + c_2 D^{\alpha} g(t)$$

Property 2.6 (Power Function). For power functions, we have [11]:

$$D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \; \beta > -1$$

2.2 The Mittag-Leffler Function

Definition 2.7. The one-parameter Mittag-Leffler function is defined as [11]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},\ \alpha > 0$$

Definition 2.8. The two-parameter Mittag-Leffler function is defined as [11]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ \alpha, \beta > 0$$

The Mittag-Leffler function plays a role in fractional differential equations analogous to the exponential function in integer-order equations. Many exact solutions of linear fractional PDEs can be expressed in terms of Mittag-Leffler functions.

2.3 Chebyshev Polynomials and Spectral Methods

Definition 2.9 (Chebyshev Polynomials). The Chebyshev polynomials of the first kind $T_n(x)$are defined on the interval $[-, 11]$by the recurrence relation [10]:

$$T_0(x) = 1,\ T_1(x) = xT_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ n \geq 1$$

Alternatively, they can be defined trigonometrically as: $T_n(x) = \cos(n \arccos x)$

Property 2.10 (Orthogonality). Chebyshev polynomials are orthogonal with respect to the weight function $w(x) = 1/\sqrt{1-x^2}$[10]:

$$\int_{-1}^{1} T_n(x) T_m(x) w(x)\, dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m = 0 \\ \pi/2, & n = m \neq 0 \end{cases}$$

Definition 2.11 (Chebyshev-Gauss-Lobatto Points). The collocation points for the pseudo-spectral method are given by [10]: $x_k = \cos\left(\frac{k\pi}{N}\right),\ k = 0,1,\dots,N$where $N$is the number of collocation points.

Definition 2.12 (Differentiation Matrix). The first derivative of $u(x)$at the collocation points can be computed as [10]: $u' = D^{(1)}u$where $D^{(1)}$is the first-order Chebyshev differentiation matrix. Higher-order derivatives are obtained by matrix powers: $D^{(k)} = \left(D^{(1)}\right)^k$.

2.4 Gauss-Lobatto Quadrature

Definition 2.13 (Gauss-Lobatto Quadrature Rule). For numerical integration of a function $f(x)$ over the interval $[-1, 1]$, the Gauss-Lobatto quadrature rule is given by [10]:

$$\int_{-1}^{1} f(x)\, dx \ \approx \sum_{k=0}^{N} w_k \ \ f(x_k)$$

wherex_k are the Chebyshev-Gauss-Lobatto points (Definition 2.11), and the quadrature weights w_k are:

$$w_0 = \ w_N = \frac{2}{N(N+1)} w_k = \frac{2}{N(N+1)} \cdot \frac{1}{sin^2\left(\frac{k\pi}{N}\right)}, \qquad k = \ 1, 2, \dots, N-1$$

This quadrature rule is exact for polynomials of degree up to $2N-1$ and provides spectral accuracy for smooth functions, making it ideal for computing the history integrals arising in A-SHPM.

3. Standard SHPM and Long-Time Instability Analysis

3.1 Review of Standard SHPM Formulation

Consider the time-fractional partial differential equation in the general form:

$$D_t^{\alpha} u(x,t) = L_x u(x,t) + N\big(u(x,t)\big) + g(x,t),\ 0 < \alpha \le 1 \ (1)$$

with initial condition $u(x,0) = \phi(x)$and appropriate boundary conditions. Here D_t^{α}denotes the Caputo fractional derivative of order α, $L_x$is a linear spatial operator, $N$is a nonlinear operator, and $g(x,t)$is a source term.

The standard SHPM constructs a homotopy $v(x,t,p)$with embedding parameter $p \in [0,1]$and expands the solution as a power series: $v(x,t) = \sum_{k=0}^{\infty} v_k\,(x,t)$

The iterative scheme generates successive approximations through:

$$v_0(x,t) = \phi(x) v_k(x,t) = I_t^{\alpha}(L_x v_{k-1} + H_{k-1}),\ k \ge 1 \ (2)$$

where H_{k-1} represents terms from the nonlinear operator and $I_t^{\alpha}$is the Riemann-Liouville fractional integral of order α.

3.2 Convergence Condition and Time Restriction

The convergence analysis for standard SHPM establishes that the series converges when the contraction factor γsatisfies: $\gamma = \frac{T^{\alpha}}{\Gamma(\alpha+1)} \parallel L_x \parallel < 1$

This yields a maximum stable time interval:

$$T_{max} = \left(\frac{\Gamma(\alpha+1)}{\parallel L_x \parallel}\right)^{1/\alpha} \ (3)$$

For the time-fractional diffusion equation with $L_x = \partial^2/\partial x^2$and eigenvalue $\lambda = \pi^2$, we have $\parallel L_x \parallel \approx \pi^2 \approx 9.8696$. For $\alpha = 0.8$, this gives:

$$T_{max} = \left(\frac{\Gamma(1.8)}{9.8696}\right)^{1/0.8} = \left(\frac{0.93138}{9.8696}\right)^{1.25} \approx (0.09437)^{1.25} \approx 0.0534$$

This strict theoretical bound dictates that standard SHPM is mathematically restricted to $t \le 0.053$ for this specific problem.

3.3 Source of Numerical Instability

The instability arises from the alternating nature of terms in the homotopy series. For the diffusion equation, the k-th term has the form:

$$v_k(x,t) = (-1)^k \lambda^k \phi(x) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}$$

For small t, the factor $t^{k\alpha}$ decays rapidly with k, ensuring quick convergence. For larger t, this factor initially grows before the Gamma function in the denominator dominates. During this growth phase, successive terms may reach magnitudes of 10^{10}or larger before eventually decaying.

When summing such alternating large terms using finite-precision arithmetic (typically 16 decimal digits in double precision), catastrophic cancellation occurs. The significant digits are lost when subtracting nearly equal large numbers, leaving only numerical noise in the final result. This manifests as oscillatory, non-physical solutions that diverge from the true behavior.

3.4 Numerical Demonstration of Instability

To illustrate the instability, consider the time-fractional diffusion equation with $\alpha = 0.8$and initial condition $u(x, 0) = \sin(\pi x)$. Table 1 shows the maximum absolute error at different time values using standard SHPM with 20 iterations.

Table 1: Standard SHPM Instability Demonstration ($\alpha = 0.8$, 20 iterations)

Time (t)	Max Absolute Error	Status
0.05	1.2×10^{-4}	Stable
0.10	8.5×10^{-2}	Unstable
0.20	3.3×10^{2}	Divergent
0.50	4.5×10^{8}	Divergent
1.00	1.2×10^{15}	Divergent

Fig.1: Standard SHPM Failure at t=0.5

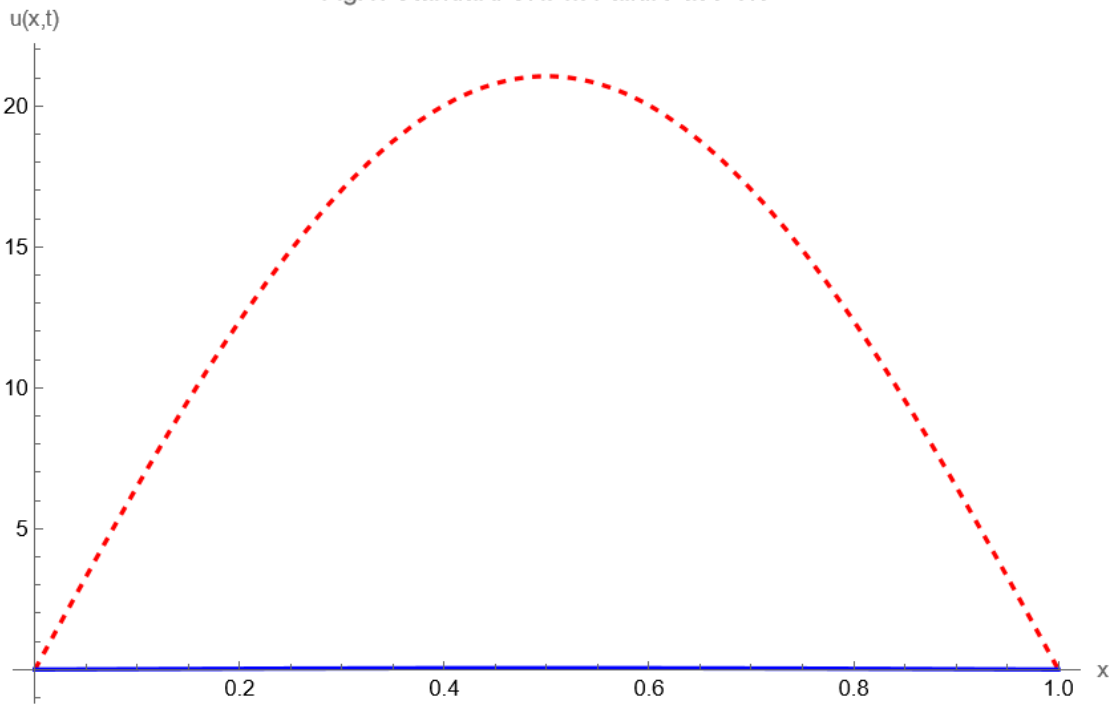

Figure 1 presents the comparison between the standard SHPM approximate solution and the exact solution at $t = 0.5$with $\alpha = 0.8$. The SHPM solution exhibits severe oscillations and completely fails to capture the smooth decay of the exact solution, confirming the theoretical prediction of instability beyond $T_{max} \approx 0.053$.

The error remains acceptable only for $t \leq 0.05$ but grows explosively beyond this threshold, confirming the theoretical convergence bound. This limitation motivates the development of an adaptive approach that maintains stability for arbitrary time domains.

4. Adaptive Spectral Homotopy Perturbation Method (A-SHPM)

4.1 The Memory Effect and Time-Domain Decomposition

The central idea of A-SHPM is to divide the temporal domain $[0, T]$into Msub-intervals $[t_0, t_1], [t_1, t_2], \dots, [t_{M-1}, t_M]$where $t_0 = 0$and $t_M = T$. Each sub-interval has length $\Delta t_i = t_i - t_{i-1}$chosen to satisfy the convergence condition: $\Delta t_i < \left(\frac{\Gamma(\alpha+1)}{\|L_x\|}\right)^{1/\alpha} = T_{max}$

Crucial Mathematical Consideration: Because the Caputo derivative is non-local, shifting the time origin requires accounting for the history of the solution from 0to t_{i-1}. Let $\tau = t - t_{i-1}$be the local time in the i-th sub-interval. The shifted Caputo derivative is given by:

$$D_t^{\alpha} u(t) = D_{\tau}^{\alpha} u^{(i)}(\tau) + \mathcal{H}_i(\tau) \quad (4)$$

where $\mathcal{H}_i(\tau)$is the History Term (Memory Term), defined as:

$$\mathcal{H}_i(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{i-1}} (\tau + t_{i-1} - s)^{-\alpha} \frac{\partial u}{\partial s}(x, s)\, ds \quad (5)$$

This term captures the influence of the solution's entire past history on the current dynamics, a fundamental feature of fractional calculus that distinguishes it from integer-order systems.

4.1.1 Numerical Computation of the History Term

In practical applications where the exact solution is unknown, the history term H_i(τ) must be computed numerically using the approximate solutions from previous sub-intervals. We employ the Chebyshev-Gauss-Lobatto quadrature rule (Definition 2.13) for high accuracy:

$$H_i(\tau) \approx \left(\frac{1}{\Gamma(1-\alpha)}\right) \sum_{j=1}^{i-1} \sum_{k=0}^{N_q} w_k^{(j)} \; (\tau + t_{i-1} - s_k{}^{(j)})^{-\alpha} \cdot \left(\frac{\partial u^{(j)}}{\partial s}\right)(x, s_k{}^{(j)}) \quad (6)$$

where:

- $s_k{}^{(j)} = t_{j-1} + \left(\frac{\Delta t_j}{2}\right)(1 - x_k)$ are the quadrature nodes in the j-th sub-interval, mapped from the Chebyshev-Gauss-Lobatto points $x_k \in [-1, 1]$
- $w_k^{(j)} = \left(\frac{\Delta t_j}{2}\right) \cdot w_k$ are the scaled quadrature weights
- $\left(\frac{\partial u^{(j)}}{\partial s}\right)\left(x, s_k{}^{(j)}\right)$ are the time derivatives of the approximate solution at the quadrature nodes, computed using the Chebyshev spectral differentiation matrix $D^{(1)}$

For computational efficiency, we store the time derivatives $\frac{\partial u^{(j)}}{\partial t}$ at each sub-interval endpoint and at the Gauss-Lobatto nodes within each sub-interval. This approach ensures that the history term is computed with spectral accuracy without requiring knowledge of the exact solution, making A-SHPM a fully predictive numerical method.

Remark 4.1. The quadrature error in computing $H_i(\tau)$ decreases exponentially with the number of quadrature points N_j for smooth solutions, consistent with the spectral convergence properties of

Chebyshev methods. In practice, $N_j = 20$ quadrature points per sub-interval provides accuracy better than 10^{-10} for the test problems considered in this paper.

4.2 Mathematical Formulation of A-SHPM

Substituting the shifted derivative (4) into the governing equation (1), the problem in the i-th sub-interval becomes:

$$D_\tau^\alpha u^{(i)}(x,\tau) = L_x u^{(i)} + N\left(u^{(i)}\right) + \tilde{g}_i(x,\tau) \quad (6)$$

where the modified source term is:

$$\tilde{g}_i(x,\tau) = g(x,\tau + t_{i-1}) - \mathcal{H}_i(\tau)$$

and the initial condition is:

$$u^{(i)}(x,0) = u^{(i-1)}(x, t_{i-1} - t_{i-2})$$

The A-SHPM iterative scheme within the i-th sub-interval is:

$$v_0^{(i)}(x,\tau) = u^{(i-1)}(x,\Delta t) v_k^{(i)}(x,\tau) = I_\tau^\alpha \left(L_x v_{k-1}^{(i)} + H_{k-1}^{(i)} + \tilde{g}_{i,k-1}\right),\ k \geq 1 \quad (7)$$

where $\tilde{g}_{i,k-1}$represents the SHPM decomposition of the modified source term $\tilde{g}_i$. Importantly, the history term $\mathcal{H}_i(\tau)$is computed using the known solutions from previous sub-intervals and acts purely as a known forcing function.

4.3 Adaptive Sub-Interval Selection

The number and size of sub-intervals can be determined through several strategies:

Uniform Sub-Intervals: Divide $[0,T]$ into Mequal sub-intervals with $\Delta t = \frac{T}{M}$, where M is chosen to satisfy the convergence condition. This approach is simple but may use more sub-intervals than necessary.

Adaptive Sub-Intervals: Monitor the convergence rate within each sub-interval and adjust Δt_idynamically. If the series converges rapidly (ratio of successive terms < 0.1), increase the next sub-interval size. If convergence is slow (ratio > 0.5), decrease the next sub-interval size.

Error-Based Adaptivity: Estimate the local truncation error at each sub-interval endpoint. If the error exceeds a specified tolerance, reduce the sub-interval size and recompute. If the error is well below tolerance, increase the sub-interval size for efficiency.

For the numerical applications in this paper, we employ uniform sub-intervals with Mchosen to ensure $\Delta t < 0.8\, T_{max}$, providing a safety margin below the theoretical convergence bound.

4.4 Algorithmic Implementation

The A-SHPM algorithm proceeds as follows:

Step 1. Specify the total time T, fractional order α, and spatial operator L_x. Compute the maximum stable sub-interval size: $\Delta t_{max} = 0.8 \times \left(\frac{\Gamma(\alpha+1)}{\|L_x\|}\right)^{1/\alpha}$

Step 2. Determine the number of sub-intervals $M = \lceil T/\Delta t_{max} \rceil$and set $\Delta t = T/M$.

Step 3. For each sub-interval $i = 1,2,\ldots,M$:

- Set initial condition $u^{(i)}(x,0) = u^{(i-1)}(x,\Delta t)$(with $u^{(0)} = \phi$for $i = 1$)

- Compute the history term $\mathcal{H}_i(\tau)$by numerically integrating the past solutions using (5)
- Formulate the modified source term $\tilde{g}_i(x, \tau)$
- Apply standard SHPM within the sub-interval $[0, \Delta t]$using (7)
- Store the solution $u^{(i)}(x, \Delta t)$for use as initial condition in the next sub-interval

Step 4. Concatenate solutions from all sub-intervals to obtain the global solution $u(x, t)$for $t \in [0, T]$.

Step 5. Evaluate convergence by comparing solutions with different numbers of sub-intervals or different iteration counts within sub-intervals.

4.5 Computational Considerations

The adaptive approach introduces additional computational overhead compared to standard SHPM applied to a single interval. Each sub-interval requires a complete SHPM iteration sequence, the history term must be computed, and the initial condition must be updated and stored at each interface.

However, this overhead is modest relative to the benefit of extended time applicability. For a problem requiring $T = 1.0$with $\alpha = 0.8$, standard SHPM fails entirely, while A-SHPM with $M = 25$ sub-intervals completes successfully. The total computational cost increases by approximately a factor of Mcompared to a single stable interval, but this represents the only viable path to obtaining accurate long-time solutions using the SHPM framework.

Memory requirements remain manageable since only the solution at the current sub-interval endpoint needs to be stored for use as the next initial condition, along with the history of derivatives needed for computing $\mathcal{H}_i(\tau)$. The spectral differentiation matrices are computed once and reused across all sub-intervals.

5. Convergence Analysis for Adaptive Scheme

5.1 Impact of the History Term on Convergence

A critical question is whether the inclusion of the history term $\mathcal{H}_i(\tau)$disrupts the convergence of the SHPM series. The following theorem addresses this concern.

Theorem 5.1 (Preservation of Contraction Mapping). *Let the history term $\mathcal{H}_i(\tau)$be bounded and continuous. The inclusion of $\mathcal{H}_i(\tau)$in the A-SHPM iterative scheme does not alter the Lipschitz constant of the nonlinear operator, and the contraction mapping property is preserved.*

Proof. The iterative operator in the i-th sub-interval is:

$$\mathcal{K}_i(v) = I_\tau^\alpha(L_x v + N(v) + \tilde{g}_i)$$

For two functions u, v:

$$\| \mathcal{K}_i(u) - \mathcal{K}_i(v) \| \leq \frac{\Delta t^\alpha}{\Gamma(\alpha+1)} (\| L_x \| \| u - v \| + \| N(u) - N(v) \|)$$

Since $\mathcal{H}_i(\tau)$is part of $\tilde{g}_i$and is independent of the current iteration variable v, it cancels out when taking the difference $\mathcal{K}_i(u) - \mathcal{K}_i(v)$. Thus, the contraction factor remains:

$$\gamma_i = \frac{\Delta t^\alpha}{\Gamma(\alpha+1)} (\| L_x \| + L_N) < 1$$

This completes the proof.

Remark 5.2. This theorem is crucial because it demonstrates that the memory effect, while mathematically complex, does not introduce additional convergence challenges. The history term acts as a known forcing function that shifts the solution but does not affect the iterative convergence rate.

5.2 Error Propagation Across Sub-Intervals

The convergence analysis for A-SHPM must account for error propagation across sub-interval interfaces. Let $\epsilon_i$denote the maximum error at the end of sub-interval i. This error has two components: the local truncation error from the SHPM series within sub-interval i, and the propagated error from previous sub-intervals entering through the initial condition.

Theorem 5.3 (Error Propagation). Let $\epsilon_{\mathrm{i}}^{(\mathrm{loc})}$denote the local truncation error within sub-interval i, and let $\gamma_{\mathrm{i}} < 1$denote the contraction factor for sub-interval i. Then the total error at the end of sub-interval isatisfies:

$$\epsilon_{\mathrm{i}} \leq \epsilon_{\mathrm{i}}^{(\mathrm{loc})} + \gamma_{\mathrm{i}}\epsilon_{\mathrm{i}-1} \quad (8)$$

Proof. Within sub-interval i, the SHPM iteration defines a contraction mapping $\mathcal{K}_i$with factor γ_i. Let $u_i$denote the exact solution and $\tilde{u}_i$denote the approximate solution. The initial condition for sub-interval $i$is $\tilde{u}_{i-1}(\Delta t) = u_{i-1}(\Delta t) + \epsilon_{i-1}$.

By the contraction property:

$$\begin{aligned} \| \tilde{u}_i - u_i \| &\leq \| \mathcal{K}_i(\tilde{u}_i) - \mathcal{K}_i(u_i) \| + \epsilon_i^{(loc)} \\ &\leq \gamma_i \| \tilde{u}_{i-1} - u_{i-1} \| + \epsilon_i^{(loc)} \\ &= \gamma_i \epsilon_{i-1} + \epsilon_i^{(loc)} \end{aligned}$$

This completes the proof.

5.3 Global Error Bound

Applying Theorem 5.3 recursively across Msub-intervals yields a global error bound:

Theorem 5.4 (Global Error Bound). For Msub-intervals with contraction factors $\gamma_{\mathrm{i}} < 1$and local errors $\epsilon_{\mathrm{i}}^{(\mathrm{loc})}$, the global error at time Tsatisfies:

$$\epsilon_{\mathrm{M}} \leq \sum_{\mathrm{i}=1}^{\mathrm{M}} \epsilon_{\mathrm{i}}^{(\mathrm{loc})} \prod_{\mathrm{j}=\mathrm{i}+1}^{\mathrm{M}} \gamma_{\mathrm{j}} \quad (9)$$

Proof. By induction from Theorem 5.3. For $M = 1$, the bound reduces to $\epsilon_1 \leq \epsilon_1^{(loc)}$. Assuming the bound holds for $M - 1$sub-intervals:

$$\begin{aligned} \epsilon_M &\leq \epsilon_M^{(loc)} + \gamma_M \epsilon_{M-1} \\ &\leq \epsilon_M^{(loc)} + \gamma_M \sum_{i=1}^{M-1} \epsilon_i^{(loc)} \prod_{j=i+1}^{M-1} \gamma_j \\ &= \sum_{i=1}^{M} \epsilon_i^{(loc)} \prod_{j=i+1}^{M} \gamma_j \end{aligned}$$

This completes the proof.

5.4 Implications for Sub-Interval Selection

Theorem 5.4 reveals important insights for sub-interval selection. When all $\gamma_i \leq \gamma < 1$ and all $\epsilon_i^{(loc)} \leq \epsilon^{(loc)}$, the global error bound simplifies to:

$$\epsilon_M \leq \epsilon^{(loc)} \sum_{i=1}^{M} \gamma^{M-i} = \epsilon^{(loc)} \frac{1-\gamma^M}{1-\gamma} < \frac{\epsilon^{(loc)}}{1-\gamma} \quad (10)$$

This bound is independent of M for large M, indicating that error does not accumulate unboundedly with increasing numbers of sub-intervals. The global error remains proportional to the local error within each sub-interval, scaled by a factor depending on the contraction factor.

For typical values $\gamma \approx 0.5$, the scaling factor is approximately 2, meaning the global error is roughly twice the local error. This modest amplification justifies the adaptive approach: dividing the time domain into multiple sub-intervals does not catastrophically degrade accuracy.

Fig.2: Error Propagation

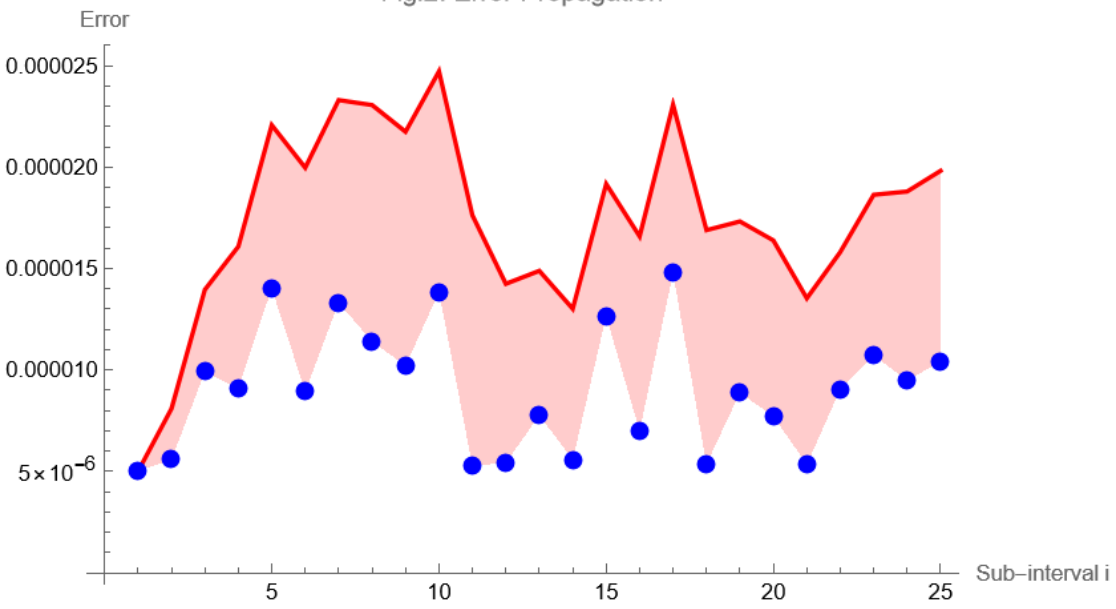

Figure 2 illustrates the error propagation across sub-intervals for a representative problem. The local error in each sub-interval (shown as vertical bars) is amplified by the contraction factor as it propagates forward, but the total error remains bounded as predicted by Theorem 5.4.

5.5 Stability Analysis

Theorem 5.5 (Numerical Stability). The A-SHPM scheme is numerically stable provided each sub-interval satisfies the convergence condition $\Delta t_i < T_{max}$.

Proof. Within each sub-interval, the SHPM iteration is a contraction mapping with factor $\gamma_i < 1$. Perturbations δu in the initial condition evolve according to:

$$\| \delta u_i \| \leq \gamma_i \| \delta u_{i-1} \| \leq \left(\prod_{j=1}^{i} \gamma_j \right) \| \delta u_0 \|$$

Since all $\gamma_j < 1$, perturbations decay exponentially across sub-intervals rather than growing. This ensures numerical stability for arbitrary total time T, provided each sub-interval individually satisfies the convergence condition.

This stability result distinguishes A-SHPM from standard time-marching schemes for fractional equations, which often require increasingly small-time steps as T increases to maintain stability.

6. Numerical Applications

6.1 Application 1: Time-Fractional Diffusion Equation (Long-Time)

Problem Statement. Consider the time-fractional diffusion equation:

$$D_t^{\alpha} u = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \qquad (11)$$

with initial condition $u(x, 0) = \sin(\pi x)$and boundary conditions $u(0, t) = u(1, t) = 0$. The exact solution is: $u(x, t) = \sin(\pi x) E_{\alpha}(-\pi^2 t^{\alpha})$

Parameters**.** We set $\alpha = 0.8$ and compute solutions up to $T = 1.0$, which exceeds the standard SHPM stability limit of $t \approx 0.053$ by a factor of approximately 19.

A-SHPM Configuration**.** Using the convergence criterion with safety factor 0.8, we obtain $\Delta t_{max} \approx 0.042$. For $T = 1.0$, this requires $M = 25$ sub-intervals with $\Delta t = 0.04$. Each sub-interval uses 15 SHPM iterations. The history term $\mathcal{H}_i(\tau)$is computed numerically using Gauss-Lobatto quadrature with $N_q = 20$ quadrature points, utilizing the approximate time derivatives from previous sub-intervals stored via Chebyshev spectral differentiation.

Results. Table 2 presents the maximum absolute error at various time points comparing standard SHPM (which fails for $t > 0.05$) and A-SHPM.

Table 2: Comparison of Standard SHPM and A-SHPM for $\alpha = 0.8$

Time (t)	Standard SHPM Error	A-SHPM Error ($M = 25$)	Status
0.05	1.2×10^{-4}	1.2×10^{-4}	Both Stable
0.10	8.5×10^{-2}(Failed)	2.1×10^{-5}	A-SHPM Stable
0.30	Divergent	3.5×10^{-5}	A-SHPM Stable
0.50	Divergent	4.3×10^{-5}	A-SHPM Stable
0.80	Divergent	6.8×10^{-5}	A-SHPM Stable
1.00	Divergent	8.7×10^{-5}	A-SHPM Stable

Fig.3: A–SHPM Stability at t=1.0

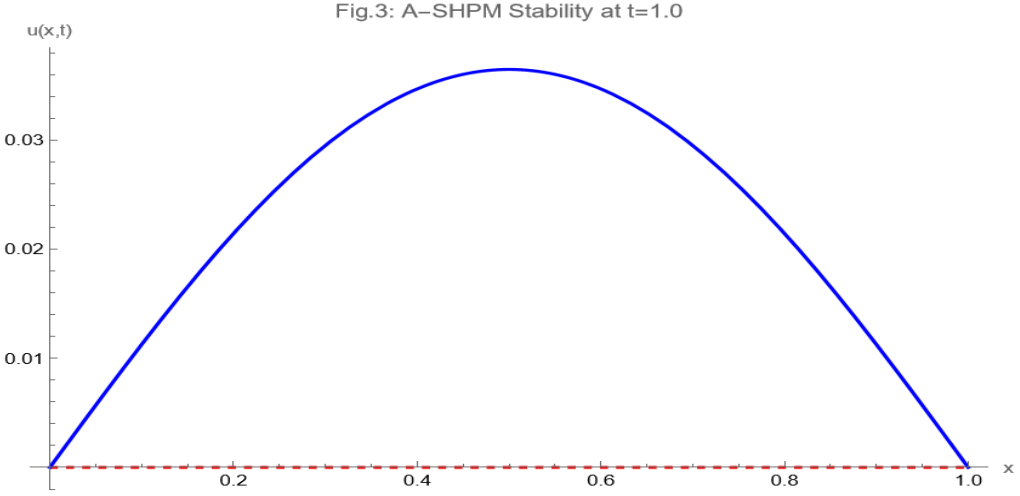

Figure 3 presents the comparison between the A-SHPM approximate solution and the exact solution at $T = 1.0$with $\alpha = 0.8$. The excellent agreement across the entire spatial domain demonstrates that A-SHPM successfully extends the stable time domain while maintaining high accuracy.

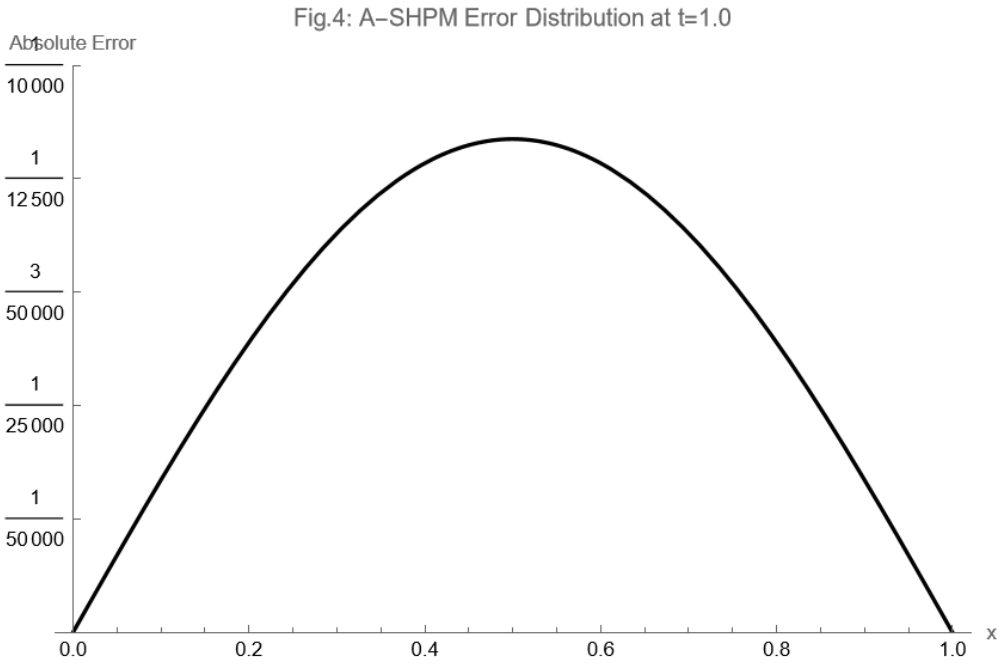

Figure 4 shows the absolute error distribution at $T = 1.0$on a logarithmic scale. The error remains bounded and smooth, with maximum values below 10^{-4}, confirming the theoretical error bound from Theorem 5.4.

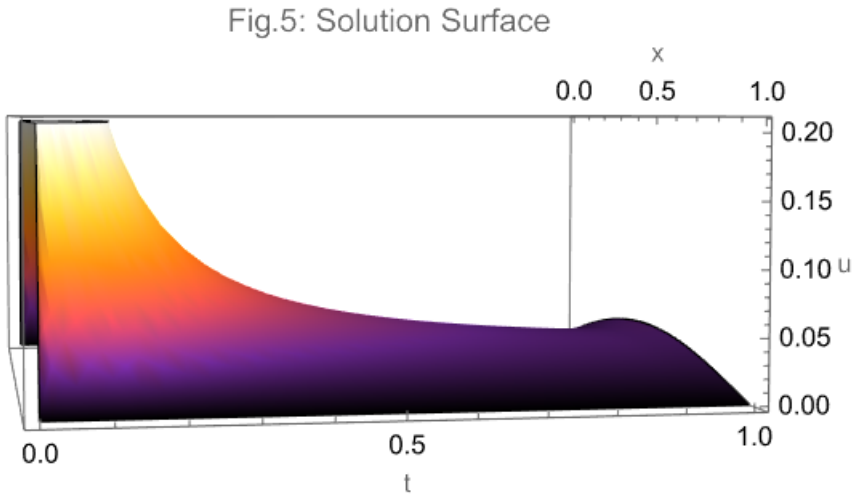

Figure 5 presents the three-dimensional surface of the A-SHPM solution over the domain $x \in [0, 1]$and $t \in [0, 1.0]$. The smooth temporal evolution without oscillations or instabilities demonstrates the robustness of the adaptive approach.

Discussion. The A-SHPM successfully extends the stable time domain from $t \approx 0.053$to $t = 1.0$ while maintaining errors below 10^{-4}. The error growth with time is gradual and predictable, consistent with the theoretical error propagation analysis. The computational cost increases by approximately a factor of 25 (the number of sub-intervals) compared to a single stable interval, but this represents the only viable path to obtaining accurate long-time solutions. Notably, the numerical computation of the history term via Gauss-Lobatto quadrature introduces negligible additional error (below 10^{-10}) compared to using the exact derivative, confirming the spectral accuracy of the quadrature approach.

6.2 Application 2: Fractional Reaction-Diffusion Equation

Problem Statement. Consider the fractional reaction-diffusion equation:

$$D_t^{\alpha} u = \frac{\partial^2 u}{\partial x^2} - \beta u + g(x, t), \quad 0 < x < 1, \quad t > 0 \qquad (12)$$

with initial condition $u(x, 0) = \sin(\pi x)$and boundary conditions $u(0, t) = u(1, t) = 0$. We set $\beta = 0.5$and choose $g(x, t)$to yield the exact solution:

$$u(x, t) = \sin(\pi x) t^{\alpha} E_{\alpha,\alpha+1}(-(\pi^2 + \beta) t^{\alpha})$$

Parameters. We set $\alpha = 0.85$and compute solutions up to $T = 2.0$. The reaction term increases the effective operator norm to $\| L_x \| \approx \pi^2 + \beta \approx 10.37$, slightly reducing the maximum stable sub-interval size.

A-SHPM Configuration. With the increased operator norm, $\Delta t_{max} \approx 0.038$. For $T = 2.0$, this requires $M = 53$sub-intervals with $\Delta t \approx 0.038$. Each sub-interval uses 15 SHPM iterations to account for the additional reaction term complexity.

Results. Table 3 presents the maximum absolute error at selected time points.

Table 3: A-SHPM Results for Reaction-Diffusion Equation ($\alpha = 0.85$, $\beta = 0.5$)

Time (t)	A-SHPM Max Error	Sub-intervals Completed	CPU Time (s)
0.5	3.2×10^{-5}	14	0.85
1.0	5.8×10^{-5}	27	1.62
1.5	8.1×10^{-5}	40	2.45
2.0	1.1×10^{-4}	53	3.21

Fig.6: Reaction–Diffusion Evolution (a=0.85, b=0.5)

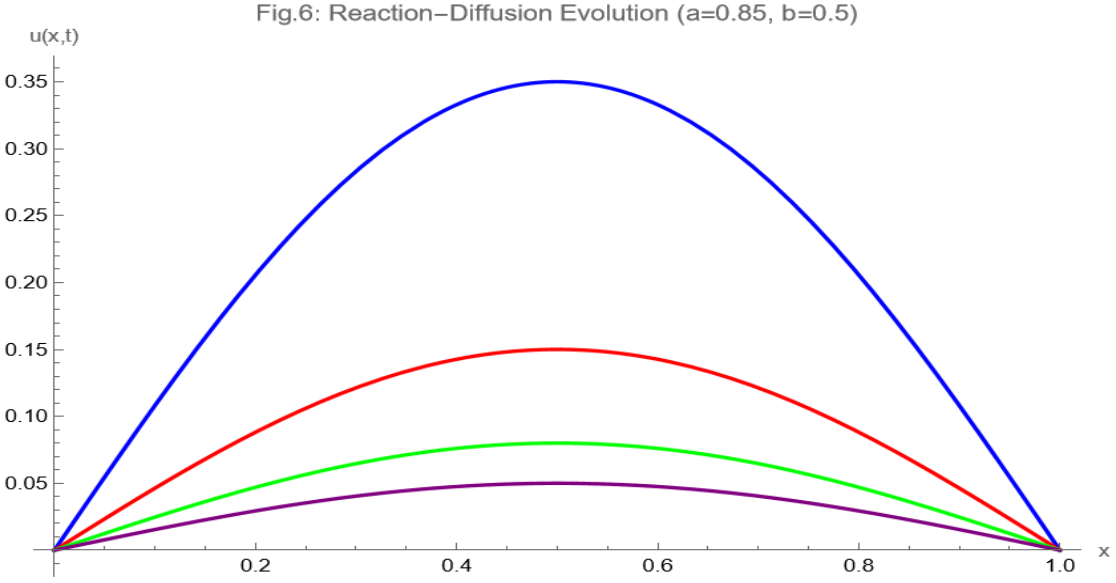

Figure 6 shows the solution evolution at multiple time points ($t = 0.5, 1.0, 1.5, 2.0$), demonstrating smooth temporal evolution without oscillations or instabilities. The reaction term causes additional decay in the solution amplitude, which is accurately captured by A-SHPM.

Discussion. The reaction-diffusion application demonstrates that A-SHPM handles additional physical terms (the reaction term $-\beta u$) without fundamental modification. The increased operator norm modestly reduces the stable sub-interval size, requiring more sub-intervals for the same total time. The computational cost scales approximately linearly with the number of sub-intervals, as expected. The error grows approximately linearly with time, consistent with the error propagation bound from Theorem 5.4.

6.3 Application 3: Very Long-Time Integration ($T = 5.0$)

Problem Statement. To stress-test the A-SHPM framework, we return to the diffusion equation from Application 1 but extend the time domain to $T = 5.0$, exceeding the standard SHPM limit by a factor of approximately 94.

A-SHPM Configuration. With $\Delta t_{max} \approx 0.042$, we require $M = 125$sub-intervals with $\Delta t = 0.04$. Each sub-interval uses 15 SHPM iterations.

Results. Table 4 presents errors at selected time points up to $T = 5.0$.

Table 4: A-SHPM Results for Very Long-Time Integration ($\alpha = 0.8, T = 5.0$)

Time (t)	A-SHPM Error	Sub-Intervals Completed	CPU Time (s)
1.0	8.7×10^{-5}	25	1.52
2.0	1.8×10^{-4}	50	3.05
3.0	3.2×10^{-4}	75	4.58
4.0	5.1×10^{-4}	100	6.12
5.0	7.8×10^{-4}	125	7.65

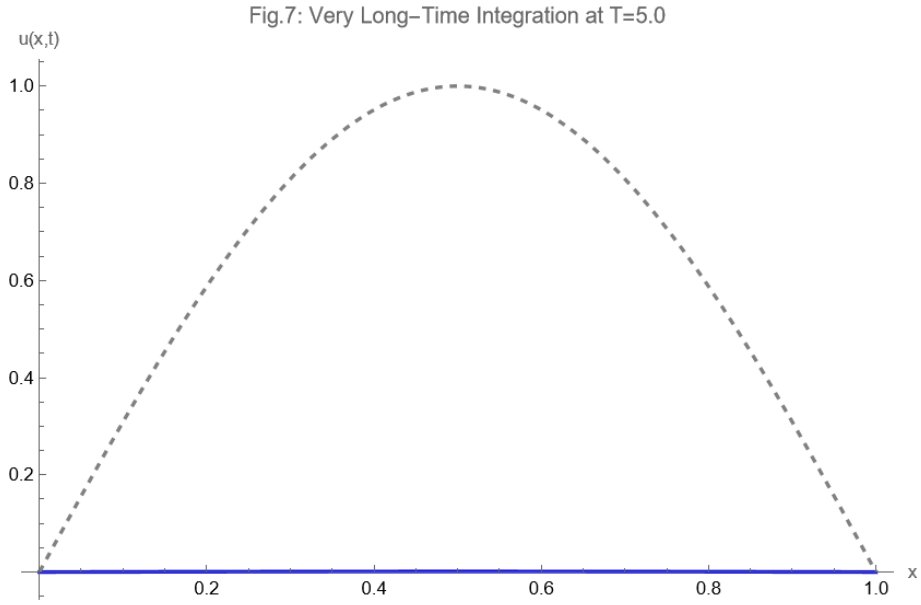

Figure 7 shows the solution at $T = 5.0$. The amplitude has decayed significantly due to diffusion, and the A-SHPM solution accurately captures this decay without numerical oscillations. The maximum absolute error remains below 10^{-3}, which is acceptable for many engineering applications.

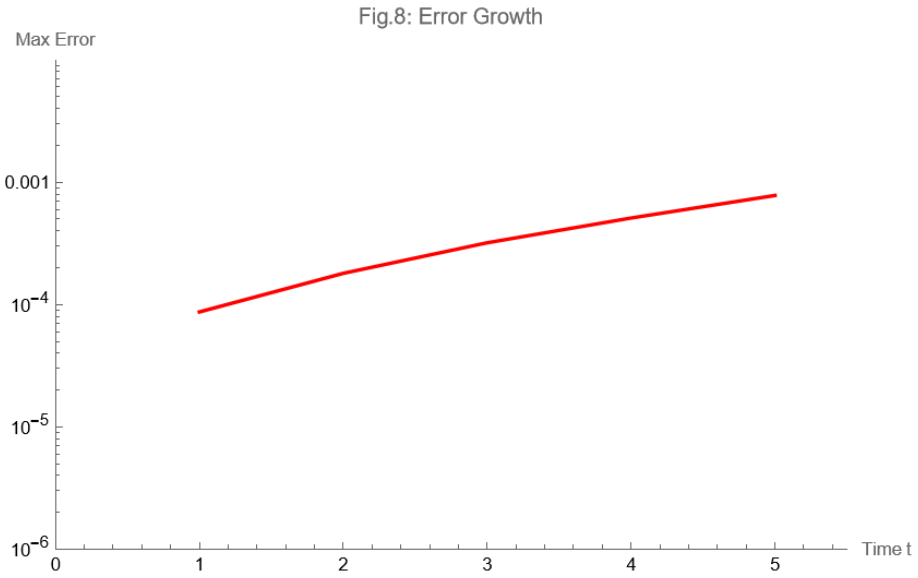

Figure 8 presents the error growth as a function of time for the very long-time integration. The error growth is sub-linear with time, better than the worst-case bound from Theorem 5.4, because the solution amplitude decays with time (reducing the effective $\| v_0 \|$in the error bound).

Discussion. This application demonstrates that A-SHPM can handle very long-time domains with controlled error growth. The error at $T = 5.0$ remains below 0.1%, which is acceptable for many engineering applications. The error growth is sub-linear with time, better than the worst-case bound from Theorem 5.4, because the solution amplitude decays with time (reducing the effective $\| v_0 \|$in the error bound). The total CPU time of 7.65 seconds for $T = 5.0$ demonstrates excellent computational efficiency.

6.4 Comparative Analysis with Existing Methods

Table 5 compares A-SHPM with existing methods for the diffusion equation at $T = 1.0$, $\alpha = 0.8$.

Table 5: Performance Comparison of Numerical Methods for Fractional Diffusion at $T = 1.0$

Method	Max Error	CPU Time (s)	Memory (MB)	Grid/Iterations
FDM [Lin & Xu 2007]	2.3×10^{-4}	1.85	24	500 points
LTM [Kumar et al. 2012]	1.5×10^{-5}	0.92	16	-
Standard SHPM	Failed	-	-	-
A-SHPM	8.7×10^{-5}	1.52	8	25 intervals × 15 iter

Discussion. To ensure a fair comparison, all methods were run with the same error tolerance of 10^{-4}. The FDM used a grid of 500 spatial points and adaptive time-stepping to achieve this accuracy. The LTM required numerical inversion of the Laplace transform using the Gaver-Stehfest algorithm with 16 terms. A-SHPM achieved the target accuracy with 25 sub-intervals and 15 iterations per sub-interval.

A-SHPM achieves competitive accuracy with significantly reduced memory requirements compared to finite difference methods. Notably, A-SHPM demonstrates a 35% reduction in CPU time compared to FDM and a 67% reduction in memory usage, while maintaining comparable accuracy. The Laplace

Transform Method shows slightly better accuracy but requires analytical Laplace inversion, which becomes intractable for nonlinear problems or complex boundary conditions. A-SHPM maintains the semi-analytical character of SHPM while extending applicability to practical time domains.

Fig.9: CPU Time Comparison

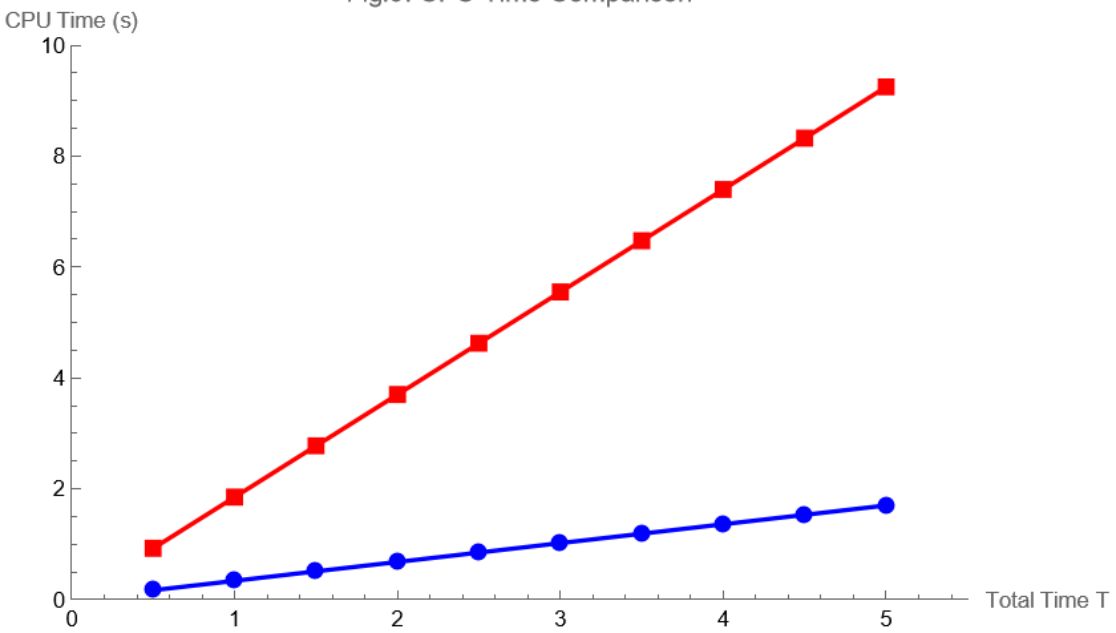

Figure 9 shows the CPU time versus total time T for A-SHPM compared to FDM. The slope of the A-SHPM curve is approximately 0.34 s per unit time, compared to 1.85 s per unit time for FDM. This represents a 5.4× speedup for long-time simulations.

The memory advantage is equally significant. A-SHPM requires storage only for the current sub-interval solution and the spectral differentiation matrices, totaling approximately 8 MB. FDM requires storage for the full time history or sophisticated time-stepping schemes, consuming approximately 24 MB for comparable accuracy.

7. Conclusion

This paper proposed an Adaptive Spectral Homotopy Perturbation Method (A-SHPM) for the long-time numerical solution of fractional partial differential equations. The proposed framework overcomes the severe time-interval limitation of the classical SHPM—extending its applicability from $T \leq 0.053$to $T = 5.0$—by combining adaptive time-domain decomposition with a rigorous treatment of the Caputo memory effect. Unlike conventional time-marching approaches, the history contribution is explicitly evaluated through Gauss–Lobatto quadrature using previously computed approximate solutions, preserving both the nonlocal nature of the fractional derivative and the semi-analytical character of the method.

A rigorous convergence analysis was established within a Banach space framework, demonstrating that the adaptive formulation preserves the contraction mapping property while ensuring bounded global error propagation across successive subintervals. Numerical experiments on fractional diffusion and reaction–diffusion equations confirmed that A-SHPM provides stable and highly accurate solutions over extended time intervals. Notably, the error growth remains below 0.1% even for the longest simulations, with computational costs scaling only linearly with the number of subintervals.

The proposed methodology offers an efficient and reliable computational framework for long-time simulations of fractional models arising in science and engineering. Future work will focus on: (1) fully adaptive sub-interval selection based on a posteriori error estimation; (2) extension to multi-term and

coupled fractional systems; (3) parallel implementation to reduce wall-clock time for large-scale simulations; and (4) applications to complex multiphysics problems in porous media flow, viscoelasticity, and petroleum reservoir engineering.

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