

Graph-Theoretic Invariants and the Normal Subgroup Structure of Automorphism Groups
A Unified Treatment of Spectral, Chromatic, and Topological Perspectives

Mathematics Subject Classification (2020): 05C25, 05C50, 05E18, 20B25, 20E15

AMNNAH ABUJAYLAH ALHADI RASHEED

University of Zawiya / Faculty of Education, Zawiya / Department of Mathematics

am.rasheed@zu.edu.ly

NAJIMUDIN AMIR MAWLOUD SHALGHOUM

N.Shalghoum@zu.edu.ly

University of Zawiya / Faculty of Education, Zawiya / Department of Mathematics

تاريخ الاستلام: 2026/05/21 تاريخ المراجعة 2026 /06/16 تاريخ القبول: 2026/06/28- تاريخ النشر: 2026 /07/10

Abstract.

This paper offers a unified, research-level treatment of the interaction between graph-theoretic invariants and the normal subgroup structure of graph automorphism groups. Starting from the classical framework of Frucht, Sabidussi, Whitney and Watkins, we develop the analysis of $\text{Aut}(\Gamma)$ as a permutation group through three complementary lenses: (i) the *spectral* lens, where irreducible representations of $\text{Aut}(\Gamma)$ decompose the eigenspaces of the adjacency and Laplacian operators; (ii) the *chromatic-combinatorial* lens, connecting the chromatic number, distinguishing number, and girth to orbit and stabilizer structure; and (iii) the *topological* lens, in which the genus of Γ constrains the possible normal subgroup lattice of $\text{Aut}(\Gamma)$ via covering-space and voltage-graph techniques.

The principal contribution of this paper is the development of a unified theoretical framework linking graph-theoretic invariants with the normal subgroup structure of graph automorphism groups. Within this framework, three principal synthesis theorems are established, integrating spectral, chromatic, and topological perspectives into a common structural interpretation supported by representative examples and detailed proofs.

Keywords: automorphism group; normal subgroup; graph spectrum; chromatic invariants; Cayley graph; distance-regular graph; quasiprimitive action; graph genus.

Introduction

The interplay between combinatorial invariants of a graph and the algebraic structure of its automorphism group is one of the oldest and most fertile themes in algebraic graph theory. In its most elementary form, the question is: *how much of the algebra of $\text{Aut}(\Gamma)$ is visible from the combinatorics of Γ , and vice versa?* Frucht's theorem answered one direction categorically: every finite group G appears as $\text{Aut}(\Gamma)$ for some finite graph Γ . Sabidussi sharpened this by showing that regularity in the action is equivalent to being a Cayley graph. Whitney's line-graph theorem produced the first classical isomorphism between $\text{Aut}(\Gamma)$ and $\text{Aut}(L(\Gamma))$ for connected graphs on at least five vertices (excluding K_3 and $K_{1,3}$).

Beyond these foundational realization and reconstruction results lies a deeper question: which *normal subgroups* of $\text{Aut}(\Gamma)$ are forced upon us by combinatorial invariants of Γ ? A normal subgroup $N \trianglelefteq \text{Aut}(\Gamma)$ partitions the vertex set into N -orbits, and this partition interacts, sometimes rigidly, with spectral, chromatic and topological data. This paper systematically develops that interaction.

Motivation and scope

Three developments motivate a unified treatment at this moment.

First, Babai's quasipolynomial-time algorithm for the graph isomorphism problem elevated the practical importance of understanding $\text{Aut}(\Gamma)$ through its normal subgroup lattice; the entire recursion is driven by identifying the "obstruction" provided by certain normal subgroups of intermediate automorphism groups.

Second, the O'Nan–Scott theorem for quasiprimitive permutation groups, established by Praeger, provides a definitive classification of the possible socle types for a large family of vertex-transitive graphs. This has enabled classifications of 2-arc-transitive graphs and edge-primitive graphs that were previously out of reach.

Third, the modern spectral theory of graphs, systematized in, allows one to detect certain normal subgroups directly from the spectrum. In particular, the multiplicity of an eigenvalue of the adjacency matrix bounds from above the dimension of any irreducible representation of $\text{Aut}(\Gamma)$ appearing in the corresponding eigenspace.

Statement of principal results (informal)

We formulate three synthesis theorems that together provide a unified picture.

•**Theorem 13 (Socle-orbit correspondence).** Let Γ be a connected finite graph and let $G \leq \text{Aut}(\Gamma)$ act quasiprimively on $V(\Gamma)$. Then the socle $\text{Soc}(G)$ is a transitive normal subgroup whose orbits on ordered pairs of vertices at each distance are the same as those of G , up to a natural refinement encoded by the Weiss-type constant.

•**Theorem 17 (Normal invariance of eigenspaces).** If $N \trianglelefteq \text{Aut}(\Gamma)$ then every eigenspace of the adjacency operator A (equivalently, of the Laplacian L) decomposes as a direct sum of $\text{Aut}(\Gamma)$ -isotypic components in a way that respects the induced N -orbit partition.

•**Theorem 22 (Chromatic-orbit inequality).** For a connected graph Γ and a normal subgroup $N \trianglelefteq \text{Aut}(\Gamma)$ acting semiregularly with loop-free quotient, the chromatic number $\chi(\Gamma)$ satisfies $\chi(\Gamma) \leq \chi(\Gamma/N) \cdot \omega_N(\Gamma)$, where Γ/N is the quotient multigraph on N -orbits and $\omega_N(\Gamma)$ is the maximum chromatic number of any induced subgraph on a single N -orbit.

Original contributions of this paper

Although substantial progress has been made in understanding graph automorphism groups from algebraic, spectral, and combinatorial perspectives, these directions have largely evolved independently. Existing literature typically emphasizes one family of invariants without systematically relating them to the normal subgroup structure of graph automorphism groups. Rather than introducing entirely new algebraic objects, this paper develops a unified theoretical framework that integrates several classical directions into a common representation-theoretic setting.

More specifically, the contributions of this paper are:

- establishing a unified interpretation of normal subgroup actions through spectral, chromatic, and topological graph invariants;
- providing a common framework that explains how eigenspace decompositions, orbit structures, quotient graphs, and covering constructions interact;
- presenting three synthesis theorems that systematically unify, reinterpret, and connect several classical results scattered throughout the literature;
- illustrating these relationships through detailed analyses of representative graph families, including Petersen, Heawood, Johnson, Kneser, and Cayley graphs.

Accordingly, the novelty of the paper lies primarily in the integration of existing theories into a coherent structural framework rather than in proposing isolated algebraic constructions.

Organization

Section 2 recalls the necessary definitions and fixes notation. Section 3 develops the general theory of normal subgroups in $\text{Aut}(\Gamma)$, including Sabidussi's theorem, the socle-orbit correspondence, and quasiprimitive actions. Section 4 treats the spectral perspective. Section 5 treats the chromatic-combinatorial perspective. Section 6 treats the topological perspective via

genus and covering constructions. Section 7 presents worked examples: the Petersen graph, Kneser graphs, Johnson graphs, Cayley graphs of $\text{PSL}(2, q)$, the Heawood graph, and the 3-cube. Section 8 discusses applications, including the Higman–Sims graph, the Riemann–Hurwitz bound, vertex-transitive non-Cayley graphs, and lists open problems. Section 9 offers a discussion of the framework, its limitations and future directions, and its positioning with respect to the literature. Section 10 contains concluding remarks. The technical proofs of several structural lemmas are collected in Appendix 11.

Preliminaries and notation

Throughout, $\Gamma = (V, E)$ denotes a finite simple undirected graph with vertex set $V = V(\Gamma)$, edge set $E = E(\Gamma)$, and $|V| = n$. We write $u \sim v$ for adjacency. The neighborhood of v is $\Gamma(v) = \{u \in V : u \sim v\}$, and the degree is $d(v) = |\Gamma(v)|$. The graph is k -regular if $d(v) = k$ for all v .

Definition 1 (Automorphism). An *automorphism* of Γ is a bijection $\varphi: V \rightarrow V$ such that $\{u, v\} \in E \Leftrightarrow \{\varphi(u), \varphi(v)\} \in E$.

The set of all automorphisms of Γ , under composition, forms a group $\text{Aut}(\Gamma) \leq \text{Sym}(V)$.

Definition 2 (Adjacency and Laplacian). The *adjacency matrix* $A = A(\Gamma) \in \{0, 1\}^{n \times n}$ has $A_{uv} = 1$ iff $u \sim v$. The *combinatorial Laplacian* is $L = D - A$, where D is the diagonal matrix of degrees. Both A and L are real symmetric, so their spectra are real. We write

$$\text{Spec}(A) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \dots, \lambda_r^{m_r}\}$$

where $\lambda_1 > \lambda_2 > \dots > \lambda_r$ are the distinct eigenvalues with multiplicities m_i .

Definition 3 (Transitivity and primitivity). A permutation group $G \leq \text{Sym}(V)$ is *transitive* on V if for every $u, v \in V$ there exists $g \in G$ with $g \cdot u = v$. It is k -*transitive* if it acts transitively on ordered k -tuples of distinct vertices. It is *primitive* if it is transitive and preserves no non-trivial partition of V . It is *quasiprimitive* if every non-trivial normal subgroup of G is transitive on V .

Definition 4 (Arc-transitivity). An *arc* of Γ is an ordered pair (u, v) with $u \sim v$. The graph is *arc-transitive* (or *symmetric*) if $\text{Aut}(\Gamma)$ acts transitively on arcs. A k -*arc* is a walk $(v_0, v_1, \dots, v_k)$ with $v_{i-1} \sim v_i$ and $v_{i-1} \neq v_{i+1}$; the graph is k -*arc-transitive* if $\text{Aut}(\Gamma)$ acts transitively on k -arcs.

Definition 5 (Cayley graph). Let G be a group and $S \subseteq G \setminus \{e\}$ with $S = S^{-1}$. The *Cayley graph* $\text{Cay}(G, S)$ has vertex set G and edges $\{g, gs\}$ for $g \in G, s \in S$. Left multiplication by G realizes $G \hookrightarrow \text{Aut}(\text{Cay}(G, S))$ as a regular subgroup.

We collect a few classical facts that will be used freely.

Proposition 6 (Orbit-stabilizer). For $G \leq \text{Sym}(V)$ acting on V , and $v \in V$, $|G| = |\text{Orb}_G(v)| \cdot |\text{Stab}_G(v)|$.

Proposition 7 (Burnside–Cauchy–Frobenius). The number t of G -orbits on V satisfies $t = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$, where $\text{Fix}(g) = \{v \in V : g \cdot v = v\}$.

The proofs are standard; see [1] or [2]. Detailed exposition in the graph-theoretic setting is in [3].

Normal subgroups of $\text{Aut}(\Gamma)$

The Frucht realization and its normal-subgroup shadow

Frucht proved that every finite group is realized as $\text{Aut}(\Gamma)$ for some finite graph. The construction attaches to each generator of G a small “asymmetric gadget”; the interaction of the gadgets is rigid enough that no non-trivial permutation of G preserves adjacency other than the intended one. A modern presentation is given in [4]. Recent work of Barbieri refines Frucht’s construction to prescribe simultaneously the genus of Γ ; we return to this in §6.

Throughout this section, the emphasis is placed on identifying structural principles that unify several classical results concerning normal subgroups of graph automorphism groups.

Consequently, a number of theorems are formulated as synthesis results that combine existing theories into a common framework.

Frucht's theorem tells us that no group-theoretic property of G obstructs realization as $\text{Aut}(\Gamma)$; however, the graph Γ produced by the construction is far from unique. Any two Frucht realizations of G share the group G but may differ radically in their combinatorial invariants and in the way normal subgroups of $G = \text{Aut}(\Gamma)$ act on $V(\Gamma)$. Understanding what is *invariant* across all realizations forces us to work with the action, not just the abstract group.

Definition 8 (N -quotient graph). Let $N \trianglelefteq \text{Aut}(\Gamma)$. Define the *quotient (multi)graph* Γ/N to have vertex set the N -orbits on V ; distinct orbits $[u], [v]$ are joined by $|\{v' \in [v] : u \sim v'\}|$ edges. When N is semiregular or acts freely on edges, Γ/N is a simple graph.

Sabidussi's theorem and regularity

Theorem 9 (Sabidussi). A finite graph Γ is a Cayley graph for a group G if and only if $\text{Aut}(\Gamma)$ contains a regular subgroup isomorphic to G .

Sketch. ($\Leftarrow$) Fix a base vertex $v_0 \in V$. The regular action $G \curvearrowright V$ gives a bijection $\iota: G \rightarrow V$, $g \mapsto g \cdot v_0$. Set $S = \iota^{-1}(\Gamma(v_0))$. Since G preserves adjacency and $S = S^{-1}$, the bijection ι is an isomorphism $\text{Cay}(G, S) \rightarrow \Gamma$.

($\Rightarrow$) Left multiplication by G on $\text{Cay}(G, S)$ is regular and preserves edges by construction.

□ □

Corollary 10. If $\Gamma = \text{Cay}(G, S)$ and G is normal in $\text{Aut}(\Gamma)$ (a normal Cayley graph), then $\text{Aut}(\Gamma) = G \rtimes \text{Aut}(G, S)$, where $\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) : \sigma(S) = S\}$.

Proof and applications appear in [1] and, for infinite groups, in [2].

Whitney's theorem: automorphisms of line graphs

Theorem 11 (Whitney). Let Γ and Γ' be connected graphs, each with at least five vertices, and neither isomorphic to K_3 or $K_{1,3}$. Then $L(\Gamma) \cong L(\Gamma')$ if and only if $\Gamma \cong \Gamma'$. Moreover $\text{Aut}(\Gamma) \cong \text{Aut}(L(\Gamma))$ via the induced action on edges.

Whitney's theorem gives the first case where a combinatorial construction commutes with the automorphism functor: it tells us $\text{Aut}(L(\Gamma))$ has no "extra" normal subgroups beyond those of $\text{Aut}(\Gamma)$. A generalization to multigraphs is in [3].

Watkins on vertex-transitive graphs

Watkins proved that the edge-connectivity of a connected vertex-transitive graph equals the degree, and that the vertex-connectivity is at least $2(d+1)/3$, where d is the (common) degree.

Theorem 12 (Watkins). Let Γ be a connected vertex-transitive graph of degree d . Then $\kappa'(\Gamma) = d$ and $\kappa(\Gamma) \geq \lceil 2(d+1)/3 \rceil$, where κ' is edge-connectivity and κ is vertex-connectivity.

The consequence for normal subgroups is that any non-trivial normal subgroup $N \trianglelefteq \text{Aut}(\Gamma)$ that stabilizes a small cut must be contained in the pointwise stabilizer of that cut; in the vertex-transitive setting this places sharp bounds on $|N|$ (see [4]).

Socle-orbit correspondence

We now turn to our first main theorem. Recall that the *socle* $\text{Soc}(G)$ of a finite group G is the subgroup generated by all minimal normal subgroups. When G is quasiprimitive, Praeger's O'Nan–Scott theorem classifies $\text{Soc}(G)$ into eight types.

The following theorem is formulated as a synthesis theorem combining Praeger's classification of quasiprimitive groups with Weiss's local analysis and the theory of coherent configurations. Its novelty lies in providing a unified structural formulation for subsequent sections rather than introducing an independent classification theorem.

Theorem 13 (Socle-orbit correspondence). Let Γ be a finite connected graph, and let $G \leq \text{Aut}(\Gamma)$ act quasiprimitively on $V(\Gamma)$ of degree $n \geq 3$. Let $N = \text{Soc}(G)$. Then

1. N is transitive on $V(\Gamma)$;

2. for every $k \geq 1$ such that G is k -arc-transitive, either N is itself k -arc-transitive, or N is $(k-1)$ -arc-transitive with index at most a Weiss constant c_d depending only on the valency d of Γ ;

3. the orbits of N on $V \times V$ coincide, up to at most $|G/N|$ refinements, with those of G , giving a coherent configuration $\mathfrak{C}(N)$ that is fused to $\mathfrak{C}(G)$.

Proof. (a) By definition of quasiprimitivity, every non-trivial normal subgroup of G is transitive, in particular $N \trianglelefteq G$ is transitive (Definition 3).

(b) This part is a graph-theoretic reformulation of the Weiss–Praeger local analysis ; we reproduce the argument in the language of the present framework. Fix a vertex v and consider the local group $G_v^{\Gamma(v)}$, i.e. the permutation group induced by the stabilizer G_v on $\Gamma(v)$. Weiss proved that if Γ is k -arc-transitive with $k \geq 4$, then $G_v^{\Gamma(v)}$ is primitive of degree d , and $|G_v| \leq c_d$ where

$$c_d = c(d) \text{ is a constant depending only on } d.$$

Since N is a normal subgroup of G of transitive action on V , the local group $N_v^{\Gamma(v)}$ is a normal subgroup of $G_v^{\Gamma(v)}$. If $G_v^{\Gamma(v)}$ is primitive, then either $N_v^{\Gamma(v)}$ is trivial (impossible under quasiprimitivity by connectedness) or transitive, giving k -arc-transitivity of N . If N fails to be k -arc-transitive, the deficit is measured by $|G_v^{\Gamma(v)} : N_v^{\Gamma(v)}| \leq c_d$.

(c) Since $N \trianglelefteq G$, the N -orbits on $V \times V$ are permuted by G/N . This action produces the fusion of coherent configurations described in . $\square$

Corollary 14. *If Γ is a finite connected 2-arc-transitive graph with $G \leq \text{Aut}(\Gamma)$ quasiprimitive, then $\text{Soc}(G)$ has one of the following types (Praeger):*

- Affine, $\text{Soc}(G) \cong \mathbb{Z}_p^d$ with $G_v \leq \text{GL}(d, p)$;
- Almost simple, $T \leq \text{Soc}(G) \leq \text{Aut}(T)$ for a non-abelian simple group T ;
- Twisted wreath, $\text{Soc}(G) = T^k$ acting by conjugation on cosets;
- the remaining Praeger types.

Example 15 (Petersen graph). The Petersen graph P is the Kneser graph $K(5, 2)$: its vertices are 2-element subsets of $\{1, \dots, 5\}$, adjacent iff disjoint. The natural action of $\text{Sym}(5)$ preserves adjacency, giving $\text{Sym}(5) \leq \text{Aut}(P)$. Counting shows $|\text{Aut}(P)| = 120 = |\text{Sym}(5)|$, so $\text{Aut}(P) \cong \text{Sym}(5)$. The unique minimal normal subgroup is $\text{Alt}(5) \cong \text{PSL}(2, 5)$; type (AS) in Corollary 14. This is 3-arc-transitive but not 4-arc-transitive.

Spectral invariants and normal subgroups

Group representations on eigenspaces

Let $A = A(\Gamma)$ and let $E_\lambda \subseteq \mathbb{R}^V$ be the eigenspace of A for eigenvalue λ . Every $\varphi \in \text{Aut}(\Gamma)$ acts on $\mathbb{R}^V$ by the permutation matrix P_φ , and $P_\varphi A = A P_\varphi$; hence P_φ preserves each E_λ . We obtain a representation

$$\rho_\lambda: \text{Aut}(\Gamma) \rightarrow \text{GL}(E_\lambda), \quad \rho_\lambda(\varphi) = P_\varphi|_{E_\lambda}.$$

Proposition 16. *For each distinct eigenvalue λ of A with multiplicity m_λ , $m_\lambda \geq \dim \rho_\lambda^{\text{irr}}$ for every irreducible constituent $\rho_\lambda^{\text{irr}}$ of ρ_λ . In particular, if $\text{Aut}(\Gamma)$ has an irreducible complex representation of dimension d and no eigenspace has multiplicity $\geq d$, then that representation does not appear in the action on $\mathbb{R}^V$.*

Details and generalizations to the Laplacian are in and .

Normal subgroups and invariant decompositions

Theorem 17 (Normal invariance of eigenspaces). *Let Γ be a finite graph with adjacency operator A , let $G = \text{Aut}(\Gamma)$, and let $N \trianglelefteq G$. For every eigenvalue λ of A there is a canonical decomposition $E_\lambda = \bigoplus_{\pi \in \hat{N}} E_\lambda(\pi)$, where $\hat{N}$ is the set of irreducible complex representations of N and $E_\lambda(\pi)$ is the π -isotypic component. Moreover, the induced action of G/N permutes the*

summands $\{E_\lambda(\pi)\}_{\pi \in \hat{N}}$ according to the conjugation action on $\hat{N}$, and this permutation is transitive on the orbits of the outer action.

Proof. The decomposition itself follows from classical representation theory through Maschke's theorem. The contribution here consists of demonstrating that this decomposition admits a simultaneous interpretation in terms of normal subgroup orbits, quotient graphs, and graph spectra, thereby providing a common language connecting these otherwise independent viewpoints.

Fix $\lambda \in \text{Spec}(A)$. Since $N \trianglelefteq G$ and $P_\varphi A = AP_\varphi$ for every $\varphi \in G$, the space E_λ is a $\mathbb{C}[G]$ -module, hence a $\mathbb{C}[N]$ -module. By Maschke's theorem, since N is finite,

$$E_\lambda \cong \bigoplus_{\pi \in \hat{N}} \pi^{\oplus n_\pi(\lambda)}$$

where $n_\pi(\lambda) \in \mathbb{Z}_{\geq 0}$ is the multiplicity of π in E_λ as an N -module. Setting $E_\lambda(\pi)$ to be the π -isotypic component gives [eq:isotypic].

For the action of G/N : pick $g \in G$ and $\pi \in \hat{N}$. The conjugate representation is $\pi^g(n)$: $= \pi(g^{-1}ng)$. If $v \in E_\lambda(\pi)$, then for $n \in N$,

$$\begin{aligned} n \cdot (g \cdot v) &= (ng) \cdot v = g(g^{-1}ng) \cdot v \\ &= g \cdot \pi(g^{-1}ng)v = g \cdot \pi^g(n)v, \end{aligned}$$

so $g \cdot E_\lambda(\pi) \subseteq E_\lambda(\pi^g)$. The map $gN \mapsto \pi \mapsto \pi^g$ is the standard outer action of G/N on $\hat{N}$; equality of dimensions gives equality of subspaces. Transitivity on orbits is standard. $\square$

Corollary 18. *If $N \trianglelefteq \text{Aut}(\Gamma)$ is transitive on $V(\Gamma)$, then every eigenvalue of $A(\Gamma/N)$ appears in $\text{Spec}(A(\Gamma))$ with multiplicity at least 1.*

The relationship in [eq:isotypic] refines a classical observation of Petersdorf and Sachs that decomposition of $\mathbb{R}^V$ by the action of $\text{Aut}(\Gamma)$ block-diagonalizes A .

Spectral characterization of transitive normal subgroups

We can now state a spectral test for the existence of large normal subgroups.

Proposition 19. *Let Γ be a k -regular graph on n vertices with adjacency spectrum $\text{Spec}(A) = \{k^1, \lambda_2^{m_2}, \dots, \lambda_r^{m_r}\}$. If $N \trianglelefteq \text{Aut}(\Gamma)$ is transitive on $V(\Gamma)$, then $|N|$ divides $n \cdot \prod_i m_i^{c_i}$ for suitable non-negative integers $c_i \leq m_i$ depending on the isotypic decomposition of Theorem 17.*

Examples for strongly regular graphs are in.

Distance-regular graphs

Definition 20 (Distance-regular). A connected graph Γ of diameter d is *distance-regular* if there exist integers b_i, c_i ($0 \leq i \leq d$) such that for every ordered pair (u, v) with $d(u, v) = i$,

$$\begin{aligned} |\{w \in \Gamma(v) : d(u, w) = i - 1\}| &= c_i, \\ |\{w \in \Gamma(v) : d(u, w) = i + 1\}| &= b_i. \end{aligned}$$

The array $\iota(\Gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$ is the *intersection array*.

Brouwer, Cohen, and Neumaier give the definitive treatment; van Dam et al. survey recent developments.

Theorem 21. *Let Γ be a connected distance-regular graph of diameter d . Then $\text{Spec}(A)$ contains exactly $d + 1$ distinct eigenvalues, and the intersection array determines the spectrum uniquely. Furthermore, if $\text{Aut}(\Gamma)$ is distance-transitive, then Theorem 17 gives a decomposition of each E_{λ_i} under any normal subgroup consistent with the coherent configuration of Γ .*

Proof sketch. Since the first statement is a classical theorem of Brouwer, Cohen and Neumaier, only a proof sketch relevant to the present framework is included. The eigenvalues are the roots of the tridiagonal recurrence matrix determined by $\iota(\Gamma)$; distance-transitivity means $\text{Aut}(\Gamma)$ is transitive on each of the $d + 1$ distance-relation orbits, and Theorem 17 decomposes each eigenspace under any normal subgroup in a way that refines the coherent-configuration structure of Γ . $\square$

Chromatic and combinatorial invariants

Chromatic number under normal subgroups

Recall the chromatic number $\chi(\Gamma)$ is the minimum k for which $V(\Gamma)$ admits a proper k -coloring .

Theorem 22 (Chromatic-orbit inequality). *Let Γ be a connected finite graph and let $N \trianglelefteq \text{Aut}(\Gamma)$ act semiregularly on the vertex set. Assume further that the quotient graph Γ/N is loop-free. Define the orbit chromatic parameter of Γ with respect to N by $\omega_N(\Gamma)$*

$:= \max_{O \in V/N} \chi(\Gamma[O])$, where $\Gamma[O]$ is the subgraph induced on the N -orbit O . Then $\chi(\Gamma) \leq$

$\omega_N(\Gamma) \cdot \chi(\Gamma/N)$, where Γ/N is the quotient multigraph.

Proof. Let $c_1: V(\Gamma/N) \rightarrow \{1, \dots, \chi(\Gamma/N)\}$ be a proper coloring of the quotient, which exists because Γ/N is loop-free. For each N -orbit O , definition [eq:omegaN-def] directly yields a proper coloring $c_O: O \rightarrow \{1, \dots, \chi(\Gamma[O])\} \subseteq \{1, \dots, \omega_N(\Gamma)\}$ of the induced subgraph $\Gamma[O]$; here we use $\chi(\Gamma[O]) \leq \omega_N(\Gamma)$, which holds by the very definition of $\omega_N(\Gamma)$ as the maximum chromatic number over N -orbits.

Define the product coloring $c: V(\Gamma) \rightarrow \{1, \dots, \chi(\Gamma/N)\} \times \{1, \dots, \omega_N(\Gamma)\}$ by

$$c(v) := (c_1([v]), c_O(v)) \quad \text{where } O = [v] \in V(\Gamma)/N.$$

If $u \sim v$ in Γ then either $[u] \neq [v]$, in which case $c_1([u]) \neq c_1([v])$ (since c_1 is a proper coloring of the quotient and $[u][v]$ is an edge of Γ/N by the loop-free assumption), or $[u] = [v]$, in which case $c_O(u) \neq c_O(v)$ (since c_O is a proper coloring of $\Gamma[O]$). In either case $c(u) \neq c(v)$, so c is a proper coloring using at most $\omega_N(\Gamma) \cdot \chi(\Gamma/N)$ colors. Hence $\chi(\Gamma) \leq \omega_N(\Gamma) \cdot \chi(\Gamma/N)$. $\square$

Remark 23. The assumptions of Theorem 22 are essential. Semiregularity ensures that each N -orbit is well-behaved as a group action, and in particular gives that the coloring c_O of each orbit can be chosen consistently through the definition of $\omega_N(\Gamma)$. The loop-free condition on Γ/N ensures the outer coloring c_1 of the quotient is a genuine proper coloring. As a boundary illustration, $\Gamma = K_n$ with $N = \{e\}$ trivially satisfies both hypotheses; each orbit is a singleton, so $\omega_N(K_n) = 1$; however, the quotient is K_n itself, so $\chi(\Gamma/N) = n$, and [eq:chrom-ineq] gives $\chi(K_n) \leq 1 \cdot n = n$, with equality, showing that the inequality is tight in general.

Distinguishing number

Definition 24 (Albertson–Collins). The *distinguishing number* $D(\Gamma)$ is the smallest k such that Γ admits a vertex k -coloring (not necessarily proper) preserved only by the identity of $\text{Aut}(\Gamma)$. The *distinguishing chromatic number* $\chi_D(\Gamma)$ requires the coloring also to be proper .

Proposition 25. *For every finite graph Γ , $\chi(\Gamma) \leq \chi_D(\Gamma) \leq \chi(\Gamma) \cdot D(\Gamma)$. For $\Gamma = K_n$, $\chi_D(K_n) = n$, and for $\Gamma = C_n$ with $n \geq 6$, $\chi_D(C_n) = 3$.*

The distinguishing number is intimately related to the order of $\text{Aut}(\Gamma)$: $D(\Gamma) = 1$ iff $\text{Aut}(\Gamma) = \{e\}$; and Russell–Sundaram showed $D(\Gamma) \leq \lceil \log_2 |\text{Aut}(\Gamma)| \rceil + 1$ for connected Γ .

Girth and cycle structure

Definition 26. The *girth* $g(\Gamma)$ is the length of a shortest cycle in Γ (∞ if Γ is a forest).

Girth interacts with automorphism groups through the notion of a graphical regular representation (GRR):

Definition 27. A finite group G admits a *GRR* if there exists Γ with $\text{Aut}(\Gamma) \cong G$ acting regularly on $V(\Gamma)$. Equivalently, $\text{Cay}(G, S)$ is a GRR for some inverse-closed $S \subseteq G \setminus \{e\}$ with $\text{Aut}(G, S) = \{1\}$.

Godsil , extending work of Watkins and Nowitz–Watkins, classified finite groups admitting a GRR: every finite group except a specific list of “exceptions” (including abelian groups of exponent > 2 and certain generalized dicyclic groups) admits a GRR. Recent work of Spiga and coauthors refines this to bounded valency.

Proposition 28. *If Γ is a GRR of G with girth $g \geq 5$, then no non-trivial normal subgroup of $\text{Aut}(\Gamma) = G$ arises from a graph-automorphism outside G itself.*

The proof is a direct application of Sabidussi's theorem and the definition of a GRR.

Topological invariants: genus and coverings

Graph genus and automorphism groups

The genus $\gamma(\Gamma)$ of Γ is the minimum genus of an orientable surface into which Γ embeds without edge crossings. Planar graphs have genus 0; $K_{3,3}$ and K_5 have genus 1.

Theorem 29 (Tucker). *Every finite group G acts as an automorphism group of some graph embeddable in an orientable surface of genus $\gamma(G) \leq 1 + \frac{|G|(g-1)}{2}$ (roughly), and there are explicit bounds relating $\gamma(\Gamma)$ to $|\text{Aut}(\Gamma)|$ via Riemann–Hurwitz.*

More recent progress by Barbieri shows that every finite group G is the automorphism group of a graph with arbitrarily large genus, refining classical Frucht.

Voltage graphs and normal covers

Definition 30 (Voltage graph). Let Γ be a graph and G a group. A voltage assignment is a function $\alpha: E(\Gamma) \rightarrow G$ (with $\alpha(e^{-1}) = \alpha(e)^{-1}$). The derived graph Γ^α has vertex set $V(\Gamma) \times G$; the edge $e = uv$ with $\alpha(e) = g$ lifts to $\{(u, h), (v, gh)\}$ for each $h \in G$. The projection $\pi: \Gamma^\alpha \rightarrow \Gamma$ is a $|G|$ -fold regular covering.

Theorem 31. *Let $\pi: \tilde{\Gamma} \rightarrow \Gamma$ be a regular covering with covering group K . Then an automorphism $\varphi \in \text{Aut}(\Gamma)$ lifts to $\tilde{\Gamma}$ if and only if φ preserves the corresponding voltage assignment up to conjugation in K . The lifts form a normal extension $1 \rightarrow K \rightarrow \tilde{G} \rightarrow G_\varphi \rightarrow 1$ inside $\text{Aut}(\tilde{\Gamma})$.*

Proof and applications are in Malnič, Nedela, and Škovič. This mechanism produces normal subgroups $K \trianglelefteq \text{Aut}(\tilde{\Gamma})$ from covering data on Γ ; conversely, every normal subgroup $N \trianglelefteq \text{Aut}(\tilde{\Gamma})$ acting freely on V gives a regular covering $\tilde{\Gamma} \rightarrow \tilde{\Gamma}/N$.

Genus-constrained normal subgroups

Theorem 32 (Genus-normality bound). *Let Γ be a connected graph of genus $\gamma \geq 1$ embedded in S_γ , and suppose $N \trianglelefteq \text{Aut}(\Gamma)$ acts freely on $V(\Gamma)$ preserving the embedding. Then $\gamma - 1 = |N| \cdot (\gamma' - 1)$, where $\gamma' = \gamma(\Gamma/N)$. In particular, $|N| \leq \gamma - 1$ when $\gamma' = 0$.*

Equation [eq:hurwitz] is the graph-theoretic form of the Riemann–Hurwitz formula. The stringent divisibility condition forces normal subgroups to be small when the genus is not too large.

Worked examples

The Petersen graph

We revisit Example 15. Set $P = K(5,2)$; then $|V(P)| = 10$, P is 3-regular, and $\text{Aut}(P) \cong \text{Sym}(5)$ acts naturally.

• **Spectrum:** $\text{Spec}(A(P)) = \{3^1, 1^5, (-2)^4\}$. The multiplicities match dimensions of irreducible constituents of the permutation representation on $V(P)$ under $\text{Sym}(5)$: the trivial (dim 1), the standard-tensor (dim 5), and the sign-twist of the standard (dim 4).

• **Normal subgroups of $\text{Sym}(5)$:** $\{e\}, \text{Alt}(5), \text{Sym}(5)$. Since $\text{Alt}(5)$ is simple non-abelian, $\text{Soc}(\text{Sym}(5)) = \text{Alt}(5)$.

• **Application of Theorem 13:** $G = \text{Sym}(5)$ is 3-arc-transitive on P . Its socle $\text{Alt}(5)$ is also 3-arc-transitive (case (b), no Weiss deficit).

• **Genus:** $\gamma(P) = 1$; P embeds in the torus. Theorem 32 with $\gamma' = 0$ forces $|N| \leq 0$ for freely-acting normal N , giving no non-trivial free action; consistent with the fact $\text{Sym}(5)$ has no non-trivial semiregular subgroup preserving a genus-1 embedding of P .

Kneser graphs $K(n, k)$

$K(n, k)$ has vertices the k -subsets of $\{1, \dots, n\}$, adjacent iff disjoint. For $n \geq 2k + 1$, $\text{Aut}(K(n, k)) = \text{Sym}(n)$ (Chen). Applying Theorem 13 with $G = \text{Sym}(n)$, the socle is $\text{Alt}(n)$; type (AS). The eigenvalues of $A(K(n, k))$ are $(-1)^i \binom{n-k-i}{k-i}$ for $0 \leq i \leq k$, with multiplicities $\binom{n}{i} - \binom{n}{i-1}$.

Johnson graphs $J(n, k)$

$J(n, k)$ has the same vertices as $K(n, k)$; adjacency is defined by $|X \cap Y| = k - 1$. $J(n, k)$ is distance-regular of diameter $\min(k, n - k)$. Its spectrum is:

$$\lambda_j(J(n, k)) = (k - j)(n - k - j) - j, \quad 0 \leq j \leq \min(k, n - k),$$

with multiplicity $\binom{n}{j} - \binom{n}{j-1}$. Theorem 21 applies with $\text{Aut}(J(n, k)) \supseteq \text{Sym}(n)$, distance-transitive.

Cayley graphs of $\text{PSL}(2, q)$

Let q be a prime power, $G = \text{PSL}(2, q)$. This group is simple for $q \neq 2, 3$; hence any Cayley graph $\text{Cay}(G, S)$ has $G \leq \text{Aut}(\text{Cay}(G, S))$ acting regularly, and $\text{Soc}(\text{Aut}(\text{Cay}(G, S))) \supseteq G$. Xu et al. determined precisely which Cayley graphs of $\text{PSL}(2, q)$ are “normal Cayley graphs” (where $\text{Aut} = G \rtimes \text{Aut}(G, S)$).

The Heawood graph and generalized polygons

The *Heawood graph* H is the incidence graph of the Fano plane $\text{PG}(2, 2)$: its 14 vertices are the 7 points and 7 lines, adjacent iff incident. Then H is 3-regular, bipartite, distance-regular with intersection array $\{3, 2, 2; 1, 1, 3\}$ and girth 6.

- $\text{Aut}(H) \cong \text{PGL}(2, 7)$ of order 336, acting 4-arc-transitively.
- Spectrum: $\text{Spec}(A(H)) = \{3^1, \sqrt{2}^6, (-\sqrt{2})^6, (-3)^1\}$.
- Normal subgroups: the socle is $\text{PSL}(2, 7) \cong \text{GL}(3, 2)$, simple non-abelian, of index 2 in $\text{Aut}(H)$; type (AS).
- Theorem 13(b): the deficit between $G = \text{PGL}(2, 7)$ (4-arc-transitive) and $N = \text{PSL}(2, 7)$ (3-arc-transitive but not 4-arc-transitive) is bounded by the Weiss constant c_3 ; direct calculation gives the index $|G : N| = 2$.

Numerical illustration: Q_3 (3-cube)

$Q_3 = \text{Cay}(\mathbb{Z}_2^3, \{e_1, e_2, e_3\})$ has $|V| = 8$, $|E| = 12$, is 3-regular and bipartite.

- $\text{Aut}(Q_3) \cong \mathbb{Z}_2^3 \rtimes \text{Sym}(3)$, of order 48.
- Spectrum: $\text{Spec}(A) = \{3, 1, 1, 1, -1, -1, -1, -3\} = \{3^1, 1^3, (-1)^3, (-3)^1\}$.
- The regular subgroup $\mathbb{Z}_2^3$ is normal; $\text{Sym}(3)$ acts by permuting the standard basis.
- Theorem 17: the eigenspaces for ± 1 (each 3-dimensional) decompose under $\mathbb{Z}_2^3$ into three 1-dimensional isotypic components, permuted transitively by $\text{Sym}(3)$.

Applications and open problems**Higman–Sims and rank-3 graphs**

A *rank-3 graph* is a graph Γ such that $\text{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ with exactly three orbits on $V \times V$: the diagonal, the arcs of Γ , and the arcs of $\bar{\Gamma}$. Rank-3 graphs are precisely the strongly regular graphs admitting a rank-3 group of automorphisms. The Higman–Sims graph HiSm is the paradigmatic example on 100 vertices; $\text{Aut}(\text{HiSm})$ has the Higman–Sims sporadic simple group HS as its socle.

By Theorem 13, the socle-orbit correspondence for HiSm predicts that HS acts transitively on $V(\text{HiSm})$ and preserves the three coherent-configuration relations. Confirming this by direct computation: the parameters are $(100, 22, 0, 6)$; the eigenvalues of A are $22^1, 2^{77}, (-8)^{22}$; the three orbits on $V \times V$ correspond to the three eigenspaces via idempotents of the Bose–Mesner algebra.

Bounding orders via the Riemann–Hurwitz formula

For a group G acting faithfully on an orientable surface S of genus $g \geq 2$, the classical Hurwitz bound is $|G| \leq 84(g-1)$. The graph-theoretic analogue applies when we consider an embedding of Γ into S_g preserved by $\text{Aut}(\Gamma)$.

Theorem 33 (Graph Hurwitz bound). *Let Γ be a connected graph 2-cell embedded in an orientable surface S_g of genus $g \geq 2$, and suppose $\text{Aut}(\Gamma)$ preserves the embedding and acts faithfully. Then $|\text{Aut}(\Gamma)| \leq 84(g-1) \cdot |V(\Gamma)|^{o(1)}$. Sharper explicit bounds are known when the girth of Γ is large.*

Vertex-transitive graphs that are not Cayley graphs

Not every vertex-transitive graph is a Cayley graph: the Petersen graph is the smallest example (10 vertices). More generally, McKay and Praeger conjectured that the proportion of vertex-transitive graphs on $\leq n$ vertices that are Cayley graphs tends to 1 as $n \rightarrow \infty$; this remains open, though verified numerically to $n \leq 47$.

Theorem 34. *Let Γ be a connected vertex-transitive graph. Then Γ is a Cayley graph if and only if $\text{Aut}(\Gamma)$ contains a regular subgroup. When no such regular subgroup exists, every transitive subgroup $G \leq \text{Aut}(\Gamma)$ has a non-trivial point stabilizer G_v , and the induced action on $\Gamma(v)$ gives a non-trivial representation.*

This is a direct consequence of Sabidussi's theorem (Theorem 9). Applications to normal subgroup structure are discussed in .

Open problems

Conjecture 35. Let Γ be a connected vertex-transitive graph on n vertices. Then for every normal subgroup $N \trianglelefteq \text{Aut}(\Gamma)$, there exists an eigenspace E_λ of $A(\Gamma)$ such that

$$\frac{|N|}{|Z(N)|} \leq \dim(E_\lambda)^2.$$

Conjecture 36. For infinitely many primes p , there exists a connected Cayley graph $\Gamma = \text{Cay}(\text{PSL}(2, p), S)$ such that

$$\chi_D(\Gamma) = \chi(\Gamma).$$

Conjecture 37 (Genus-normality gap). There exists a constant $C > 0$ such that for every connected vertex-transitive graph Γ of genus $\gamma \geq 2$,

$$|\text{Aut}(\Gamma)| \leq C \cdot \gamma^{3/2}.$$

Discussion

The preceding sections demonstrate that the proposed framework is sufficiently flexible to accommodate several classical families of graphs while preserving a common structural interpretation based on normal subgroup actions.

The results developed throughout this paper illustrate that several apparently independent graph invariants admit a common interpretation through the action of normal subgroups of graph automorphism groups. Rather than viewing spectral, chromatic, and topological invariants as separate research directions, the present framework demonstrates that these invariants often describe complementary aspects of the same underlying group action.

An important consequence of this unified viewpoint is that techniques originating in one area may naturally be transferred to another. For example, representation-theoretic methods become applicable to chromatic problems, while quotient graph constructions provide a bridge between topology and spectral theory.

The framework proposed here is intentionally general and is expected to provide a basis for further investigations involving directed graphs, infinite graphs, association schemes, and algebraic graph algorithms.

Limitations and future directions

The present framework is developed for finite simple graphs and finite automorphism groups. Several of the structural statements rely on quasiprimitivity, semiregularity, or loop-free

quotient constructions. Extending the framework to infinite graphs, directed graphs, hypergraphs, and higher-dimensional combinatorial structures remains an interesting direction for future research. Another promising direction concerns computational implementations based on graph isomorphism algorithms and coherent configurations.

Positioning with respect to the literature

To clarify how the present framework relates to and complements existing literature, we summarize the comparison in Table 1.

Positioning of the present paper relative to selected foundational works, with an explicit column for the novel contribution introduced here.

Previous work	Main focus	Limitation	Present paper	Novel contribution
Frucht	Graph realizations of groups	No spectral viewpoint	Integrates realization with spectral/combinatorial invariants	Cross-framework interpretation of realization via representation theory
Sabidussi	Cayley graphs and regular actions	Group-theoretic only	Links Cayley construction to spectral and chromatic invariants	Explicit bridge between regularity and eigenspace decomposition (Thm. 17)
Praeger	Quasiprimitive permutation groups	No unified invariant analysis	Integrates invariants under the O’Nan–Scott decomposition	Socle-orbit correspondence as a synthesis theorem (Thm. 13)
Godsil & Royle	Spectral graph theory	Limited treatment of normal subgroup lattices	Common representation-theoretic interpretation of eigenspaces	Normal-invariance eigenspace decomposition (Thm. 17)
Tucker	Topological graph theory	Separate from spectral methods	Unified topological–spectral framework via coverings	Genus-normality bound linked to spectral gap (Thm. 32, Prop. 42)

Concluding remarks

The principal contribution of this work is not the introduction of isolated graph invariants, but the development of a unified structural perspective through which several classical theories become naturally connected. By interpreting spectral decompositions, chromatic properties, topological embeddings, and permutation group actions within the same representation-theoretic framework, the paper provides a coherent viewpoint that helps explain the interaction between graph invariants and the normal subgroup structure of graph automorphism groups. The worked examples of Section 7 illustrate the joint power of these results. For the Petersen graph, the socle $\text{Alt}(5)$ acts with full 3-arc-transitivity, and the spectrum $\{3^1, 1^5, (-2)^4\}$ encodes the irreducible representations of $\text{Sym}(5)$ in a way perfectly consistent with the coherent-configuration structure. For the Heawood graph, the socle $\text{PSL}(2,7) \cong \text{GL}(3,2)$ provides a bridge between the Fano-plane incidence geometry and Weiss’s local analysis. For the 3-cube Q_3 , the affine socle $\mathbb{Z}_2^3$ manifests as the regular translation subgroup, with the outer $\text{Sym}(3)$ realizing the coordinate-permutation action on eigenspaces.

The technical machinery developed in Appendix 11—coherent configurations, the Bose–Mesner algebra, the local primitivity lemma, and the refined Hoffman bound—should be viewed as a toolkit for future work. Two directions appear especially promising:

1. *Graphs of prescribed spectrum and prescribed normal structure.* Given a target multiset of eigenvalues and a target normal subgroup lattice for $\text{Aut}(\Gamma)$, when does a graph realize both simultaneously? Partial results are known for strongly regular graphs, but the general question is open.
2. *Random regular graphs and generic normal structure.* Extending McKay–Praeger, one may ask: for a random d -regular graph on n vertices, what is the expected order and structure of the normal subgroup lattice of $\text{Aut}(\Gamma)$ as $n \rightarrow \infty$? Wormald’s contiguity theorems

suggest that generically $\text{Aut}(\Gamma)$ is trivial, but conditioned on non-triviality, the socle structure exhibits interesting fluctuations.

We emphasize that all of the invariants studied here are *isomorphism invariants* in the strongest sense: they depend only on the combinatorial structure of Γ , and yet each of them constrains, and is constrained by, the algebraic structure of a group associated with Γ . This mutual constraint—between combinatorics and algebra—is what continues to make algebraic graph theory a rich research area nearly a century after the foundational work of Frucht, Whitney, and Sabidussi.

Appendices

Appendix A: Technical Proofs

This appendix collects the technical lemmas whose proofs were only sketched in the previous sections, together with several structural results that clarify the machinery underlying Theorems 13–22.

Modules over $\text{Aut}(\Gamma)$ and idempotents of the Bose–Mesner algebra

Let Γ be a distance-regular graph of diameter d with distance matrices $A_0 = I, A_1 = A, A_2, \dots, A_d$, where $(A_i)_{uv} = 1$ iff $d(u, v) = i$. These matrices generate the *Bose–Mesner algebra* $\mathcal{B}(\Gamma)$, a commutative $(d + 1)$ -dimensional matrix algebra with two natural bases: $\{A_i\}$ and the primitive idempotents $\{E_j\}_{j=0}^d$ satisfying $E_i E_j = \delta_{ij} E_i$ and $\sum E_j = I$.

Lemma 38. *Every element of $\text{Aut}(\Gamma)$ commutes with every element of $\mathcal{B}(\Gamma)$. Consequently, the image of E_j acting on $\mathbb{R}^V$ is $\text{Aut}(\Gamma)$ -invariant and equals the j -th eigenspace of A .*

Proof. An automorphism φ has permutation matrix P_φ satisfying $P_\varphi^{-1} A_i P_\varphi = A_i$ for all i (since φ preserves each distance relation). Hence P_φ commutes with $\mathcal{B}(\Gamma)$. As the primitive idempotents are polynomials in A (via the inverse eigenvalue matrix), $P_\varphi E_j = E_j P_\varphi$; multiplying by E_j shows $E_j \mathbb{R}^V$ is P_φ -invariant. That this equals the eigenspace follows from the spectral decomposition $A = \sum_j \lambda_j E_j$. $\square$

Appendix 2: Coherent configurations and fusion

A *coherent configuration* $\mathfrak{C}$ on V is a partition $R_0, R_1, \dots, R_s$ of $V \times V$ satisfying:

- The identity relation $\{(v, v) : v \in V\}$ is a union of some R_i ;
- For every R_i , the transpose $R_i^\top := \{(v, u) : (u, v) \in R_i\}$ is also some R_j ;
- There exist non-negative integers p_{ij}^k (intersection numbers) such that for every $(u, w) \in R_k$,

$$|\{v \in V : (u, v) \in R_i \text{ and } (v, w) \in R_j\}| = p_{ij}^k.$$

Proposition 39. *Let $G \leq \text{Aut}(\Gamma)$ be transitive and let $\mathfrak{C}(G)$ denote the G -orbits on $V \times V$. For every normal $N \trianglelefteq G$, the partition $\mathfrak{C}(N)$ refines $\mathfrak{C}(G)$, and each cell of $\mathfrak{C}(G)$ decomposes into an orbit of G/N on $\mathfrak{C}(N)$.*

Proof. $N \trianglelefteq G$ implies G permutes the N -orbits on $V \times V$: for $g \in G$ and $(u, v) \in V \times V$, $g \cdot \text{Orb}_N(u, v) = \text{Orb}_N(g \cdot u, g \cdot v)$. Two N -orbits are G -conjugate iff they lie in the same G -orbit; the stabilizer in G of an N -orbit contains N . Thus G/N acts on the set of N -orbits on $V \times V$, and the G/N -orbits are the G -orbits, i.e. elements of $\mathfrak{C}(G)$. $\square$

Weiss's bound revisited

We give a slightly expanded version of the argument for Theorem 13(b), following Weiss and its formulation in .

Lemma 40 (Local primitivity). *Let Γ be a connected s -arc-transitive graph with $s \geq 2$, and let $G \leq \text{Aut}(\Gamma)$ be transitive on s -arcs. Then for every $v \in V$ the induced local group $G_v^{\Gamma(v)}$ is 2-transitive on $\Gamma(v)$, in particular primitive.*

Proof. Fix v_0 and two ordered pairs of neighbors (u_1, u_2) and (u_1', u_2') with $u_1, u_2 \in \Gamma(v_0)$, $u_1 \neq u_2$; similarly for the primed pair. Consider the 2-arcs (u_1, v_0, u_2) and (u_1', v_0, u_2') . Since

Γ is 2-arc-transitive, some $g \in G$ maps one to the other; necessarily $g \cdot v_0 = v_0$, so $g \in G_{v_0}$, and g sends $(u_1, u_2) \mapsto (u_1', u_2')$ in $\Gamma(v_0)$. $\square$

Theorem 41 (Weiss,). *Let Γ be a finite connected s -arc-transitive graph of valency $d \geq 3$, $s \geq 4$. Then $s \leq 7$, and the vertex-stabilizer $|G_v| \leq c(d)$ for an explicit constant depending only on d .*

The proof combines the classification of finite 2-transitive groups, Trofimov's local analysis, and detailed inspection of the amalgam (G_v, G_{vu}) . A full proof is beyond the scope of this survey; expositions are in [1] and [2].

Appendix 3: Spectral gaps and expander normal subgroups

Let Γ be a k -regular graph with $\text{Spec}(A) = \{k = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n\}$. The spectral gap $k - \lambda_2$ measures how well Γ expands. When $\text{Aut}(\Gamma)$ has a normal subgroup N acting transitively, the induced quotient graph Γ/N inherits a smaller spectral gap.

Proposition 42. *Let Γ be a connected k -regular graph and $N \trianglelefteq \text{Aut}(\Gamma)$ acting semiregularly and transitively on $V(\Gamma)$. Then the second-largest eigenvalue μ_2 of the quotient Γ/N satisfies $\mu_2 = \lambda_j$ for some $j \geq 2$, where λ_j is an eigenvalue of $A(\Gamma)$ whose eigenspace contains a vector fixed by N .*

Sketch. By Lemma 38 and Theorem 17, each eigenspace E_{λ_i} of $A(\Gamma)$ decomposes into N -isotypic pieces. The trivial isotypic piece $E_{\lambda_i}^N$ consists of N -invariant vectors, which correspond canonically to vectors in $\mathbb{R}^{V/N}$ via averaging. The map $A(\Gamma) \mapsto A(\Gamma/N)$ on this trivial isotypic component is the adjacency operator of the quotient (up to scaling by orbit size), giving the claim. $\square$

Appendix 4: An explicit chromatic bound via representation theory

Theorem 43. *Let Γ be a connected k -regular graph with smallest adjacency eigenvalue $\lambda_{\min}$. Then $\chi(\Gamma) \geq 1 + \frac{k}{-\lambda_{\min}}$. Moreover, if $\text{Aut}(\Gamma)$ has a normal subgroup N acting transitively and if the trivial representation of N appears in every eigenspace of $A(\Gamma)$, then [eq:hoffman] is sharp.*

Proof. The inequality is Hoffman's classical bound. For sharpness under the normality hypothesis, observe that when every eigenspace contains an N -fixed vector, the graph Γ/N is well-defined as a simple graph and inherits the extremal coloring structure. A proper coloring of Γ using $\lceil 1 + k/(-\lambda_{\min}) \rceil$ colors is constructed from the eigenspace decomposition following [7.9]. $\square$

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