



A NEW SUBCLASSES OF ANALYTIC FUNCTION

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Abstract:

the main objective of this study was introduce certain new subclasses of analytic function defined in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, By using a new generalized derivative operator $D_{\lambda_1, \lambda_2}^{k, \alpha, \beta}$ and using differential subordination which are defined by means of the Hadamard product (or convolution). and In this paper we investigate various inclusion properties of these subclasses.

Keywords : Hadamard product , analytic function , inclusion properties

1- INTRODUCTION AND DEFINITIONS

Let A denote the class of functions of the form given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. which are analytic in the open unit disk U on the complex plane $\mathbb{C}$. Let $S, S^*(\alpha), C(\alpha), K(\alpha) (0 \leq \alpha < 1)$ denote the subclasses of A consisting of functions that are univalent, starlike of order α , convex of order α , and close - to- convex of order α in U , respectively.

In particular, the classes $S^*(0) = S^*$, $C(0) = C$ and $K(0) = K$ are the familiar classes of starlike, convex and close-to-convex functions in U , respectively.

DEFINITION 1 : If we have the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in T(x)$ is called a function with negative coefficient. Now $(x)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{for } n=0, x \in \mathbb{C} \setminus \{0\}. \\ x(x+1)(x+2)\dots(x+n-1) & \text{for } n \in \mathbb{N} \text{ and } x \in \mathbb{C}. \end{cases} \quad (1)$$

For numbers a, b, c other than $0, -1, -2, \dots$ the hyper geometric function ${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{(n-1)!}$,

where $(x)_n$ denote the Pochhammer symbol which defined by (1)

Note that ${}_2F_1(a, b; c; z)$ converges absolutely for all $z \in U$ so that it represents an analytic function in U , then we obtain

$${}_2F_1(a, b; c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}} \frac{z^n}{(n-1)!}.$$

Also an incomplete beta function $h(a, c, z)$ is related to Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ as $h(a, c, z) = {}_2F_1(1, a, c, z)$.

We note that $h(a, 1, z) = \frac{z}{(1-z)^a}$, where $h(2, 1, z)$ is Koebe function.

Let

$$k_a(z) = \frac{z}{(1-z)^a},$$

where a is any real number. It is easy to verify that $k_a(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(1)_{n-1}} z^n$. Thus

$$(k_a * f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(1)_{n-1}} a_n z^n.$$

Let N denotes the class of all functions ϕ which are analytic, convex and univalent in U , with normalization $\phi(0) = 1$ and $R(\phi(z)) > 0$ for $z \in U$. Making use of the principle of subordination between analytic functions, **Darus and Al-Abbadi (2009)** introduce subclasses $S^*(\phi)$, $C(\phi)$, and $K(\phi, \psi)$ of the class A for $\phi, \psi \in N$, which are defined by

$$\begin{aligned} S^*(\phi) &= \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z) \text{ in } U \right\}, \\ C(\phi) &= \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \text{ in } U \right\}, \\ K(\phi, \psi) &= \left\{ f \in A : \exists g \in S^*(\phi) \text{ s.t. } \frac{zf'(z)}{g(z)} \prec \psi(z) \text{ in } U \right\}. \end{aligned}$$

For $\phi(z) = \psi(z) = (1+z)/(1-z)$ in the definitions defined above, we have the well-known classes S^* , C and K , respectively. Furthermore, for the function classes $S^*[A, B]$ and $C[A, B]$ investigated by Janowski, it is easily seen that

$$\begin{aligned} S^*\left(\frac{1+Az}{1+Bz}\right) &= S^*[A, B] \quad (-1 \leq B < A \leq 1), \\ C\left(\frac{1+Az}{1+Bz}\right) &= C[A, B] \quad (-1 \leq B < A \leq 1). \end{aligned}$$

Definition 2: (Ruscheweyh 1975) For a function f in A , the Ruscheweyh derivative operator $D^\delta : A \rightarrow A$, is defined by

$$D^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) \quad (\delta > -1, z \in U),$$

where $*$ stands for convolution, and which implies that

If f is given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (2)$$

then we see that

$$D^m f(z) = z + \sum_{n=2}^{\infty} c(m, n) a_n z^n, \quad (3)$$

We have recently introduced a new generalized derivative operator $D_{\lambda_1, \lambda_2}^{k, \alpha, \beta}$ is defined by **A.Lsheli and M.alargat (2021)**.. Now by generate the operator which can be written as following:

$$D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) = \varphi_{\lambda_1, \lambda_2}^{k, \alpha, \beta}(z) * R^\delta f(z), (z \in U) \quad (4)$$

where $k, \delta \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\beta \geq \alpha \geq 0, \lambda_2 \geq \lambda_1 \geq 0$ and $R^\delta f(z)$ denotes the Ruscheweyh derivative operator which given by (1.7.1). Note that $\varphi_{\lambda_1, \lambda_2}^{k, \alpha, \beta}(z)$ is the analytic function given by (4)

Now if f is given by (2), then we easily find from the equality (4) that:

$$D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) = z + \sum_{n=2}^{\infty} \left[\frac{n^\alpha + (n-1)n^\alpha \lambda_1}{n^\beta + (n-1)n^\beta \lambda_2} \right]^k c(\delta, n) a_n z^n, \quad (z \in U), \quad (5)$$

where $k, \alpha, \beta \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\beta \geq \alpha \geq 0, \lambda_2 \geq \lambda_1 \geq 0, \delta \geq 0$, and

$$c(\delta, n) = \binom{n+\delta-1}{\delta} = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}.$$

Let us remind the well known Carlson-Shaffer operator $L(a, c)$ associated with the incomplete beta function $h(a, c; z)$ defined by

$$L(a, c) : A \rightarrow A,$$

$$L(a, c)f(z) = h(a, c; z) * f(z), \quad (z \in U),$$

where

$$h(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n,$$

a is any real number and $c \notin z_0^-, z_0^- = \{0, -1, -2, \dots\}$.

Now by using the new generalized derivative operator $D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}$, we introduce the following classes of analytic functions for $\phi, \psi \in N$, $\beta \geq \alpha \geq 0$, $\lambda_2 \geq \lambda_1 \geq 0$, $\delta \geq 0$ and $k, \alpha, \beta \in N_0$.

$$\begin{aligned} S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) &= \left\{ f \in A : D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) \in S^*(\phi) \right\}, \\ C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) &= \left\{ f \in A : D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) \in C(\phi) \right\}, \\ K_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi, \psi) &= \left\{ f \in A : D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) \in K(\phi, \psi) \right\}. \end{aligned}$$

In particular, we set

$$\begin{aligned} S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}\left(\frac{1+Az}{1+Bz}\right) &= S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}[A, B] \quad (-1 \leq B < A \leq 1), \\ C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}\left(\frac{1+Az}{1+Bz}\right) &= C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}[A, B] \quad (-1 \leq B < A \leq 1). \end{aligned}$$

In this paper, we investigate several inclusion properties of the classes $S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$

$C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$ and $K_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi, \psi)$. we first state some preliminary lemmas which shall be used in our investigation.

2- PRELIMINARY RESULTS

To establish our results, we recall the following:

Lemma 2.1 (Ruscheweyh & Sheil-Small 1973) If $f \in K, g \in S^*$, then for each analytic function ϕ in U ,

$$\frac{(f * \phi g)(U)}{(f * g)(U)} \subset \overline{co\phi(U)},$$

where $\overline{co\phi(U)}$ denotes the closed convex hull of $\phi(U)$.

Lemma 2.2 (Ruscheweyh 1982) Let $0 < a \leq c$. If $c \geq 2$ or $a + c \geq 3$, then the function

$$h(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \quad (z \in U),$$

belongs to the class C of convex functions.

3- INCLUSION PROPERTIES

Our main results are the following:

Theorem 3.1 $f(z) \in C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$ if and only if $zf'(z) \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$.

Proof: Consider

$$D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) = h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(z) * f(z), \quad (6)$$

where

$$h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(z) = z + \sum_{n=2}^{\infty} \left[\frac{n^\alpha + (n-1)n^\alpha \lambda_1}{n^\beta + (n-1)n^\beta \lambda_2} \right]^k c(\delta, n) z^n. \quad (7)$$

By the definition of the class $C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$ and using the well-known property of convolution

$z(f * g)'(z) = (f * zg')(z)$, we have

$$\begin{aligned}
 f \in C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) &\Leftrightarrow 1 + \frac{z (D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z))'}{(D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z))} \prec \phi(z), \\
 &\Leftrightarrow \frac{[z (D_{\lambda_1, \lambda_2}^{k, \alpha, \beta} f(z))']}{(D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z))'} \prec \phi(z), \\
 &\Leftrightarrow \frac{z [z (h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) * f(z))']}{z [h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) * f(z)]} \prec \phi(z), \\
 &\Leftrightarrow \frac{z [h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) * z f'(z)]}{h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} * z f'(z)} \prec \phi(z), \\
 &\Leftrightarrow \frac{z [D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} z f'(z)]}{D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} z f'(z)} \prec \phi(z), \\
 &\Leftrightarrow z f'(z) \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi).
 \end{aligned}$$

Theorem 3.2 Let $\phi \in N$, $\beta \geq \alpha \geq 0$, $\lambda_2 \geq \lambda_1 \geq 0$, $\delta_2 \geq \delta_1 \geq 0$ and $\delta_1, \delta_2, k, \alpha, \beta \in \mathbb{N}_0$. If $\delta_2 \geq 1$ or $\delta_2 + \delta_1 \geq 1$, then

$$S_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta}(\phi) \subset S_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta}(\phi).$$

Proof: We suppose that $f \in S_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta}(\phi)$. Then there exists an analytic function U in w with $|w(z)| < 1$ ($z \in U$) and $w(0) = 0$ such that

$$\frac{z (D_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta} f(z))'}{(D_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta} f(z))} = \phi(w(z)), \quad (z \in U). \quad (8)$$

Where ϕ is analytic and convex univalent with $\phi(0) = 1$ and $Re(\phi(z)) > 0$, ($z \in U$), we set

$$D_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta} f(z) = D_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta} f(z) * \psi_{\delta_2}^{\delta_1}(z),$$

where

$$\psi_{\delta_2}^{\delta_1}(z) = z + \sum_{n=2}^{\infty} \frac{(\delta_1 + 1)_{n-1}}{(\delta_2 + 1)_{n-1}} z^n.$$

We get

$$\begin{aligned}
 \frac{z (D_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta} f(z))'}{D_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta} f(z)} &= \frac{z [D_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta} f(z) * \psi_{\delta_2}^{\delta_1}(z)]'}{D_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta} f(z) * \psi_{\delta_2}^{\delta_1}(z)} \\
 &= \frac{\psi_{\delta_2}^{\delta_1}(z) * [z (D_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta} f(z))']}{\psi_{\delta_2}^{\delta_1}(z) * D_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta} f(z)} \\
 &= \frac{\psi_{\delta_2}^{\delta_1}(z) * \phi(w(z)) p(z)}{\psi_{\delta_2}^{\delta_1}(z) * p(z)}, \quad (9)
 \end{aligned}$$

where $p(z) = D_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta} f(z)$.

It follows from Lemma 2.2 that $\psi_{\delta_2}^{\delta_1}(z) \in C$ and it follows from the definition of the class $S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$ that $p(z) \in S^*$. Therefore applying Lemma 2.1 to (9) we obtain

$$\frac{\left\{ \psi_{\delta_2}^{\delta_1}(z) * \phi(w(z))p \right\}(U)}{\left\{ \psi_{\delta_2}^{\delta_1}(z) * P \right\}(U)} \subset \overline{co} \phi(w(U)) \subset \phi(U). \quad (10)$$

Since ϕ is analytic and convex univalent. Therefore from the definition of subordination and (10), we note that (9) is subordinate to ϕ in U and consequently $f(z) \in S_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta}(\phi)$. This completes the proof of Theorem 3.2.

Theorem 3.3 Let $\phi \in N, \beta \geq \alpha \geq 0, \lambda_2 \geq \lambda_1 \geq 0, \delta_2 \geq \delta_1 \geq 0$ and $\delta_1, \delta_2, k, \alpha, \beta \in N_0$. If then $\delta_2 + \delta_1 \geq 1$, or $\delta_2 \geq 1$

$$C_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta}(\phi) \subset C_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta}(\phi).$$

Proof: Let $f(z) \in C_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta}(\phi)$ we want to show $f(z) \in C_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta}(\phi)$. Applying Theorem 3.1 and Theorem 3.2 we observe that

$$\begin{aligned} f(z) \in C_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta}(\phi) &\Leftrightarrow zf'(z) \in S_{\lambda_1, \lambda_2, \delta_2}^{k, \alpha, \beta}(\phi), \\ &\Rightarrow zf'(z) \in S_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta}(\phi), \\ &\Leftrightarrow f(z) \in C_{\lambda_1, \lambda_2, \delta_1}^{k, \alpha, \beta}(\phi). \end{aligned}$$

Theorem 3.4: Let $\phi \in N, \beta \geq \alpha \geq 0, \lambda_2 \geq \lambda_1 \geq 0$ and $\delta_1, \delta_2, k, \alpha, \beta \in N_0$, then

$$\begin{aligned} (i) \quad S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) &\subset S_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta}(\phi) \\ (ii) \quad C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) &\subset C_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta}(\phi) \end{aligned}$$

Proof: We suppose that $f(z) \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$. Using similar arguments as in the proof of Theorem 3.2, and set

$$D_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta} f(z) = D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) * \psi_{\delta_2}^{\delta_1}(z),$$

where

$$\psi_{\lambda_2}^{\lambda_1}(z) = z + \sum_{n=2}^{\infty} \frac{n^{\alpha} + (n-1)n^{\alpha}\lambda_1}{n^{\beta} + (n-1)n^{\beta}\lambda_2} z^n.$$

We obtain

$$\begin{aligned} \frac{z (D_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta} f(z))'}{D_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta} f(z)} &= \frac{z [D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) * \psi_{\lambda_2}^{\lambda_1}(z)]'}{D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) * \psi_{\lambda_2}^{\lambda_1}(z)} \\ &= \frac{\psi_{\lambda_2}^{\lambda_1}(z) * z (D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z))'}{\psi_{\lambda_2}^{\lambda_1}(z) * D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z)} \\ &= \frac{\psi_{\lambda_2}^{\lambda_1}(z) * \phi(w(z))p(z)}{\psi_{\lambda_2}^{\lambda_1}(z) * p(z)}. \end{aligned} \quad (11)$$

Where $p(z) = D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z)$ and w is an analytic function in U with $|w(z)| < 1$ ($z \in U$) and $w(0) = 0$. It follows from the definition of the class $S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$ that $p(z) \in S^*$, and by classical results in the class of convex, the coefficients problem for convex $|a_n| \leq 1$, and here $\frac{n^{\alpha} + (n-1)n^{\alpha}\lambda_1}{n^{\beta} + (n-1)n^{\beta}\lambda_2} \leq 1$ since $\lambda_2 \geq \lambda_1 \geq 0$, and $\beta \geq \alpha \geq 0$ so we find $\psi_{\lambda_2}^{\lambda_1} \in C$. Hence it follows from applying Lemma 2.1 to (11) that

$$\frac{\{\psi_{\lambda_2}^{\lambda_1}(z) * \phi(w(z))p(z)\}(U)}{\{\psi_{\lambda_2}^{\lambda_1}(z) * p(z)\}(U)} \subset \overline{co} \phi(w(U)) \subset \phi(U). \quad (12)$$

Since ϕ is analytic and convex univalent, and from the definition of subordination and (12). Thus (11) is subordinate to ϕ in U and consequently $f(z) \in S_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta}(\phi)$.

(ii) Let $f(z) \in C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$. We shall show $f(z) \in C_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta}(\phi)$. By applying Theorem 3.1, and Theorem 3.4, we have

$$\begin{aligned} f(z) \in C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) &\Leftrightarrow zf'(z) \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi), \\ &\Rightarrow zf'(z) \in S_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta}(\phi), \\ &\Leftrightarrow f(z) \in C_{\lambda_1, \lambda_2, \delta}^{k+1, \alpha, \beta}(\phi). \end{aligned}$$

Theorem 3.5: Let $\phi \in N, \beta \geq \alpha \geq 0, \lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$ and $k, \delta \in \mathbb{N}_0$, then

$$S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) \subset S_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta}(\phi).$$

Proof: Let $f \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$. Applying the definition of the class $S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$, then set

$$D_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta} f(z) = D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z) * \psi_{\lambda_2, \lambda_3}^{k, \alpha, \beta}(z),$$

where

$$\psi_{\lambda_2, \lambda_3}^{k, \alpha, \beta}(z) = z + \sum_{n=2}^{\infty} \left[\frac{n^{\alpha} + (n-1)n^{\alpha}\lambda_2}{n^{\beta} + (n-1)n^{\beta}\lambda_3} \right]^k z^n, \quad (z \in U).$$

And by classical results in the class of convex, the coefficients problem for convex $|a_n| \leq 1$, and here

$$\left[\frac{n^{\alpha} + (n-1)n^{\alpha}\lambda_2}{n^{\beta} + (n-1)n^{\beta}\lambda_3} \right]^k \leq 1, \text{ since } \lambda_3 \geq \lambda_2 \geq 0, \text{ and } \beta \geq \alpha \geq 0.$$

so we get $\psi_{\lambda_2, \lambda_3}^{k, \alpha, \beta}(z) \in C$. After that, using the same arguments as in the Theorem 3.4 we obtain the desired result.

Theorem 3.6 Let $\phi \in N, \beta \geq \alpha \geq 0, \lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$ and $k, \delta \in \mathbb{N}_0$, then

$$S_{\lambda_2, \lambda_3, \delta}^{k, \alpha, \beta}(\phi) \subset S_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta}(\phi).$$

Proof: Let $f(z) \in S_{\lambda_2, \lambda_3, \delta}^{k, \alpha, \beta}(\phi)$, applying the definition of the class $S_{\lambda_2, \lambda_3, \delta}^{k, \alpha, \beta}(\phi)$, and set

$$D_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta} f(z) = D_{\lambda_2, \lambda_3, \delta}^{k, \alpha, \beta} f(z) * \psi_{\lambda_1, \lambda_2}^{k, \alpha, \beta}(z),$$

where

$$\psi_{\lambda_1, \lambda_2}^{k, \alpha, \beta}(z) = z + \sum_{n=2}^{\infty} \left[\frac{n^{\alpha} + (n-1)n^{\alpha}\lambda_1}{n^{\beta} + (n-1)n^{\beta}\lambda_2} \right]^k z^n, \quad (z \in U).$$

And using the same arguments as in the Theorem 3.4 we obtain the desired result.

Corollary 3.7 Let then $k, \delta \in \mathbb{N}_0$, and $\phi \in N, \beta \geq \alpha \geq 0, \lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$

$$(i) C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) \subset C_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta}(\phi).$$

$$(ii) C_{\lambda_2, \lambda_3, \delta}^{k, \alpha, \beta}(\phi) \subset C_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta}(\phi).$$

Taking

$$\phi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1, z \in U).$$

In Corollary 3.7 and Theorems 3.5 and 3.6 we have the following Corollary.

Corollary 3.8 Let $\beta \geq \alpha \geq 0, \lambda_3 \geq \lambda_2 \geq \lambda_1 \geq 0$ and $k, \delta \in \mathbb{N}_0$, then

$$S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}[A, B] \subset S_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta}[A, B], \quad (-1 \leq B < A \leq 1),$$

$$C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}[A, B] \subset C_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta}[A, B], \quad (-1 \leq B < A \leq 1),$$

$$S_{\lambda_2, \lambda_3, \delta}^{k, \alpha, \beta}[A, B] \subset S_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta}[A, B], \quad (-1 \leq B < A \leq 1),$$

$$C_{\lambda_2, \lambda_3, \delta}^{k, \alpha, \beta}[A, B] \subset C_{\lambda_1, \lambda_3, \delta}^{k, \alpha, \beta}[A, B], \quad (-1 \leq B < A \leq 1).$$

4- CONVOLUTION RESULTS

The next theorem shows that the classes $S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$ and $C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$ are invariant under convolution with convex functions.

Theorem 4.1 Let $\phi \in N, \beta \geq \alpha \geq 0, \lambda_2 \geq \lambda_1 \geq 0, k, \delta \in \mathbb{N}_0$ and let $g \in C$. Then

$$(i) f \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) \Leftrightarrow g * f \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi),$$

$$(ii) f \in C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) \Leftrightarrow g * f \in C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi).$$

Proof: (i) We begin by assuming $f \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$ and $g \in C$. In the proof we use the same techniques as in the proof of Theorem 3.2.

Let

$$\frac{z (D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z))'}{D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z)} = \phi(w(z)), \quad (z \in U)$$

and

$$p(z) = D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z).$$

Using the following equality

$$z (h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} * f)'(z) = (h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} * z f')(z),$$

from (6) we write

$$\begin{aligned} \frac{z [D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} (g * f)](z)}{D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} (g * f)(z)} &= \frac{z [h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} (z) * (g * f)(z)]}{h_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} (z) * (g * f)(z)} \\ &= \frac{g(z) * z (D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z))'}{g(z) * D_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta} f(z)} \\ &= \frac{g(z) * \phi(w(z)) p(z)}{g(z) * p(z)} \prec \phi(z). \end{aligned}$$

Consequently $g * f \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$.

(ii) Let $f \in C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi)$. Then by Theorem 3.1 and from Theorem 4.1 (i), we have

$$\begin{aligned} f \in C_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) &\Leftrightarrow z f'(z) \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) \\ &\Rightarrow g * z f'(z) \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) \\ &\Rightarrow z (g * f)' \in S_{\lambda_1, \lambda_2, \delta}^{k, \alpha, \beta}(\phi) \end{aligned}$$

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