



The Cauchy Problem in $H^1(\mathbb{R})$ for the Logarithmic Schrödinger Equation

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Abstract

This work studies the Cauchy problem for the non-linear logarithmic Schrödinger equation (LogSE) incorporating damping and an external force:

$$i u_t + \Delta u + u \ln(|u|^2) + i \gamma u = f$$

in the Sobolev space $H^1(\mathbb{R})$ with the initial condition $u(0) = u_0$ in $H^1(\mathbb{R})$. We investigate the local well-posedness (in the sense of Hadamard) of the Cauchy problem. To handle the logarithmic nonlinearity $u \ln(|u|^2)$, which presents a singularity at the origin, a rigorous functional framework based on Sobolev space properties is established. The analytical approach relies on classical fixed-point theory (Banach fixed-point theorem) to establish the existence and uniqueness of solutions.

Keywords: *Cauchy problem, Logarithmic Schrödinger equation, Damping, Sobolev spaces, Local well-posedness, Banach fixed-point theorem.*

1 Introduction

The nonlinear logarithmic Schrödinger equation, presented with damping and an external force, is written in the following form:

$$i u_t + \Delta u + u \ln(|u|^2) + i \gamma u = f.$$

(1)

Here, the unknown is $u : R_x \times R_t \rightarrow \mathbb{C}$, where u is a complex-valued function. In the following, we denote $u(t) = u(t, \cdot) \in H^1(\mathbb{R})$. The parameter $\gamma \in \mathbb{R}$ designates the damping coefficient of the system, while the function f represents the external force, assumed to belong to the space $L^2(\mathbb{R})$ and assumed to be independent of time t . This equation (LogSE) is a variant of the classical Schrödinger equation [2, 3], characterized by a logarithmic restoring force. From a physical point of view, it is introduced to model

dissipative quantum mechanics phenomena [6, 11], supersymmetric quantum mechanics [12, 13], as well as wave propagation in nonlinear optics [9, 15]. The nonlinearity introduced by the term $u \ln(|u|^2)$ presents a singularity at the origin (when $u = 0$), which requires a rigorous functional framework based on the properties of Sobolev spaces [2], and bifurcation analysis [11]. We define the Cauchy condition associated this equation by:

$$u(0) = u_0 \quad \text{in } H^1(R).$$

(2)

The aim of this article is to establish the local well-posedness (in the sense of Hadamard) of the Cauchy problem of (1) for the initial condition (2). To do this, we exploit classical techniques based on the Banach fixed-point theorem, inspired by fundamental works on the regularity and asymptotic behavior of solutions to Schrödinger equations [1, 4, 5, 7, 8, 14].

2 The Cauchy Problem

We study the Cauchy problem associated with equation 1, with x in R .

For a given u_0 and in $H^1(R)$ and $T \leq 0$, we introduce the space $E = C([0, T]; H^1(R))$ equipped with the norm

$$\|u\|_E = \sup_{[0, T]} \|u(t)\|_{H^1}.$$

For $f \in L^2(R)$, we set $F\varepsilon(u) = u \ln(|u|^2 + \varepsilon)$, equation (1) can be written in the following form

$$iu_t + \Delta u + F\varepsilon(u) + i\gamma u = f$$

(3)

Dividing equation (3) by i , we obtain

$$u_t + -i\Delta u - iF\varepsilon(u) + \gamma u = -if.$$

(4)

Thus we have

$$u_t + (-i\Delta + \gamma)u = -i(f - F\varepsilon(u)).$$

(5)

Integrating equation (5) over $[0, t]$, we obtain the mild formulation of the equation

$$u(t) = e^{-tK} u_0 - i \int_0^t e^{-(t-s)K} F(u(s)) ds,$$

(6)

Such that $K = (\gamma - i\Delta)$, $F(u(s)) = f - F\varepsilon(u(s))$.

We can write equation (6) in the following form

$$u(t) = U(t)u_0 - i \int_0^t U(t-s) F(u(s)) ds,$$

(7)

For $\gamma = 0$, $U(t)$ is a unitary group. For positive γ and positive t , $U(t)$ is a contraction semi-group with norm $\exp(-t)$. The equivalence between (1) and its Duhamel formulation (7) is classical; we refer to [10]. We set

$$G(u(t)) = U(t)u_0 - i \int_0^t U(t-s) F(u(s)) ds,$$

(8)

Proposition 1 . For a given u_0 in $H^1(R)$, there exists a local solution $u \in C([0, T]; H^1(R))$ where T is a decreasing function of $\|u_0\|_{H^1}$.

Proof. We begin by proving some lemmas regarding the logarithmic term $F\varepsilon(u)$.

Lemma 1. For u in $L^2(R)$, we have

$$\|F\varepsilon(u)\|_{L^2}^2 \leq c \|u\|_{L^2}^2 .$$

Proof. Let $u \in H^1(R)$. Thanks to the Sobolev embedding of $H^1(R)$ into $L^\infty(R)$, we have

$$\|u\|_{L^\infty} \leq c.$$

Therefore, we have

$$\begin{aligned} \|F\varepsilon(u)\|_{L^2}^2 &\leq \int |u(x) \ln(|u(x)|^2 + \varepsilon)|^2 dx \\ &\leq \int |u(x)|^2 |\ln(|u(x)|^2 + \varepsilon)|^2 dx \\ &\leq \int |u(x)|^2 (c_\varepsilon)^2 dx \\ &\leq c \|u\|_{L^2}^2 . \end{aligned}$$

Lemma 2. For u in $H^1(R)$, we have

$$\|\ln(|u|^2 + \varepsilon)\|_{H^1}^2 \leq c \|u\|_{H^1}^4 .$$

Proof.

$$\begin{aligned} \|\ln(|u|^2 + \varepsilon)\|_{H^1}^2 &= \|\ln(|u|^2 + \varepsilon)\|_{L^2}^2 + \|\partial(\ln(|u|^2 + \varepsilon))\|_{L^2}^2 \\ &\leq c_\varepsilon \| |u|^2 \|_{L^2}^2 + \left\| \frac{2 \operatorname{Re} \bar{u} \partial u}{|u|^2 + \varepsilon} \right\|_{L^2}^2 \\ &\leq c_\varepsilon \|u\|_{L^\infty}^2 \|u\|_{L^2}^2 + c_\varepsilon \|\partial u\|_{L^2}^2 \\ &\leq c_\varepsilon \|u\|_{L^2}^3 \|\partial u\|_{L^2} + c_\varepsilon \|u\|_{H^1}^2 \\ &\leq c \|u\|_{H^1}^4 . \end{aligned}$$

Lemma 3. For u in $H^1(R)$, we have

$$\|F\varepsilon(u)\|_{H^1} \leq c \|u\|_{H^1}^3.$$

Proof.

$$\begin{aligned} \|F\varepsilon(u)\|_{H^1} &\leq \|u\|_{H^1} \|u \ln(|u|^2 + \varepsilon)\|_{H^1} \\ &\leq c \|u\|_{H^1} \|u\|_{H^1}^2 \\ &\leq c \|u\|_{H^1}^3. \end{aligned}$$

Lemma 4. For u in $H^1(R)$, we have

$$\|uF\varepsilon(u)\|_{H^1} \leq c \|u\|_{H^1}^4.$$

Proof.

$$\begin{aligned} \|uF\varepsilon(u)\|_{H^1} &\leq c(\|u\|_{L^\infty} \|F\varepsilon(u)\|_{H^1} + \|u\|_{H^1} \|F\varepsilon(u)\|_{L^\infty}) \\ &\leq c\|u\|_{L^\infty} \|u\|_{H^1}^3 + \|u\|_{H^1} \|F\varepsilon(u)\|_{L^\infty} \\ &\leq c\|u\|_{L^\infty} \|u\|_{H^1}^3 + c \\ &\leq c \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \|u\|_{H^1}^3 + c \|u\|_{H^1} \\ &\leq c \|u\|_{H^1}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{1}{2}} \|u\|_{H^1}^3 + c \|u\|_{H^1} \|u\|_{H^1}^3 \\ &\leq c \|u\|_{H^1}^4. \end{aligned}$$

Lemma 5. The mapping G is well-defined from E into E .

Proof. We control G in the H^1 norm

$$\begin{aligned} \|G(u(t))\|_{H^1} &\leq \|U(t)u_0\|_{H^1} + \left\| \int_0^t U(t-s) F(u(s)) ds \right\|_{H^1} \\ &\leq \|U(t)u_0\|_{H^1} + \left\| \int_0^t U(t-s) F\varepsilon(u(s)) ds \right\|_{H^1} + \left\| \int_0^t U(t-s) f(x) ds \right\|_{H^1}, \end{aligned}$$

as $U(t)$ is contractive on $H^1(R)$, we then have

$$\|U(t)u_0\|_{H^1} \leq \|u_0\|_{H^1} \leq \frac{c_0}{4}.$$

We estimate the norm in $H^1(R)$ of

$$\int_0^t U(t-s) F\varepsilon(u(s)) ds.$$

We have

$$\begin{aligned} \left\| \int_0^t U(t-s) F\varepsilon(u(s)) ds \right\|_{H^1} &\leq \int_0^t \| U(t-s) F\varepsilon(u(s)) \|_{H^1} ds \\ &\leq \int_0^t \| F\varepsilon(u(s)) \|_{H^1} ds \\ &\leq c \int_0^t \| u(s) \|_{H^1}^3 ds \\ &\leq c C_0^3 T. \end{aligned}$$

It remains to estimate the norm in $H^1(R)$ of

$$\int_0^t U(t-s) f(x) ds,$$

We have

$$\begin{aligned} \int_0^t U(t-s) f(x) ds &= \left(\int_0^t U(t-s) ds \right) f(x) \\ &= \left(\int_0^t e^{-(t-s)K} ds \right) f(x) \\ &= K^{-1}(Id - e^{-tK}) f(x). \end{aligned}$$

Lemma 6. The operator K^{-1} is bounded from $L^2(R)$ into $H^1(R)$.

Proof. We assume that

$$\begin{aligned} Kw &= f \\ K &= (\gamma - i\Delta) \end{aligned}$$

Then

$$\gamma w - i\Delta w = f.$$

(9)

Multiplying (9) by $\bar{w}$, we integrate by taking the real part of this equation

$$\begin{aligned} \gamma \operatorname{Re} \int |w|^2 - \operatorname{Re} \int i\bar{w} \Delta w &= \operatorname{Re} \int f\bar{w}, \\ \gamma \|w\|_{L^2}^2 + \operatorname{Im} \int i\bar{w} \Delta w &= \operatorname{Re} \int f\bar{w}, \end{aligned}$$

Integrating by parts, we have

$$\gamma \|w\|_{L^2}^2 \leq \|f\|_{L^2} \|w\|_{L^2}.$$

We deduce that

$$\|w\|_{L^2}^2 \leq \frac{\|f\|_{L^2}^2}{\gamma}.$$

Multiplying (9) by $\bar{w}$, we integrate by taking the imaginary part of this equation

$$\gamma \operatorname{Im} \int |w|^2 - \operatorname{Re} \int \bar{w} \Delta w = \operatorname{Im} \int f\bar{w}$$

Integrating by parts, we have

$$\|\nabla w\|_{L^2}^2 \leq \|f\|_{L^2} \|w\|_{L^2}.$$

We deduce that

$$\|\nabla w\|_{L^2}^2 \leq \|f\|_{L^2} \frac{\|f\|_{L^2}}{\gamma}.$$

Thus

$$\|\nabla w\|_{L^2} \leq \frac{\|f\|_{L^2}}{\sqrt{\gamma}}.$$

We obtain

$$\begin{aligned} \|w\|_{H^1} &\leq \frac{\|f\|_{L^2}}{\gamma} + \frac{\|f\|_{L^2}}{\sqrt{\gamma}} \\ &\leq \frac{c_\gamma}{2} \|f\|_{L^2}. \end{aligned}$$

From which $w = K^{-1}f$, we deduce that

$$\|K^{-1}f\|_{H^1} \leq \frac{c_\gamma}{2} \|f\|_{L^2}.$$

According to this lemma, we obtain

$$\begin{aligned} \left\| \int_0^t U(t-s) f(x) ds \right\|_{H^1} &= \|K^{-1}(Id - e^{-tK}) f\|_{H^1} \\ &\leq \|K^{-1} f\|_{H^1} (\|Id\|_{H^1} + \|e^{-tK}\|_{H^1}) \\ &\leq 2 \|K^{-1} f\|_{H^1} \\ &\leq c_\gamma \|f\|_{L^2}. \end{aligned}$$

We choose C_0 such that $c_\gamma \|f\|_{L^2} \leq \frac{C_0}{4}$. We deduce that

$$\|G(u(t))\|_{H^1} \leq \frac{C_0}{2} + cC_0^3 T.$$

We choose T such that $cC_0^3 T < \frac{1}{2}$.

We obtain

$$\|G(u(t))\|_{H^1} \leq C_0.$$

Then we have

$$\|G(u)\|_E \leq \sup_{[0,T]} \|G(u(t))\|_{H^1} \leq C_0.$$

This completes the proof of Lemma 5.

Lemma 7. The mapping G is contractive on bounded subsets of E .

Proof. Let $u, v \in B(0, C_0) \subset E$ with

$$\begin{aligned}
 \|G(u(t)) - G(v(t))\|_{H^1} &= \left\| \int_0^t U(t-s) (F(u(s)) - F(v(s))) ds \right\|_{H^1} \\
 &= \left\| \int_0^t U(t-s) (-iF(u(s)) + iF(v(s))) ds \right\|_{H^1} \\
 &= \left\| \int_0^t U(t-s) (F(u(s)) - F(v(s))) ds \right\|_{H^1} \\
 &\leq \int_0^t \|F(u(s)) - F(v(s))\|_{H^1} ds \\
 &\leq \int_0^t \|u(\ln|u|^2 + \varepsilon) - v(\ln|v|^2 + \varepsilon)\|_{H^1} ds \\
 &\leq \int_0^t \|u\|_{H^1} \|\ln|u|^2 + \varepsilon - \ln|v|^2 + \varepsilon\|_{L^\infty} \|u - v\|_{H^1} ds \\
 &\leq \int_0^t \|u\|_{H^1} \|\ln|u|^2 + \varepsilon - \ln|v|^2 + \varepsilon\|_{L^\infty} \|u - v\|_{H^1} ds \\
 &\leq \int_0^t \|u\|_{H^1} \|\ln|u|^2 + \varepsilon - \ln|v|^2 + \varepsilon\|_{L^\infty} \|u - v\|_{H^1} ds \\
 &\leq cC_0^3 \int_0^t \|u - v\|_{H^1} ds.
 \end{aligned}$$

We then obtain

$$\begin{aligned}
 \|G(u) - G(v)\|_E &\leq cC_0^3 \|u - v\|_E \int_0^t ds \\
 &\leq cC_0^3 T \|u - v\|_E.
 \end{aligned}$$

We choose T such that $cC_0^3 T \leq 1$, Therefore, G is contractive on bounded subsets of E . By the fixed-point theorem, there exists a unique u_ε in E which is a mild solution of 1 such that $G(u) = u_\varepsilon$, hence a unique solution u_ε of equation 1 on $[0, T]$ with values in $B(0, C_0)$.

Passing to the limit. The sequence $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in the space $L^\infty(0, T; H^1(R))$ we have

$$\|u_\varepsilon\|_{L^\infty} \leq C.$$

By the Banach-Alaoglu theorem, we obtain $\forall T > 0$, $u_\varepsilon \rightharpoonup u$ weakly * in $L^\infty(0, T; H^1(R))$.

By the local compactness theorem, the embedding of $H^1(R)$ into $L^2_{loc}(R)$ being compact, we obtain the strong convergence $u_\varepsilon \rightarrow u$ strongly in $L^2(0, T; H^1(R))$.

$u_\varepsilon(x, t) \rightarrow u(x, t)$
a.e. on $[0, T]$.

We have

$$F(u_\varepsilon) = u_\varepsilon \ln(|u_\varepsilon|^2 + \varepsilon).$$

By continuity of the logarithm function on $]0, +\infty[$, we have

$$F(u_\varepsilon) = u_\varepsilon \ln(|u_\varepsilon|^2 + \varepsilon) \rightarrow F(u) = u \ln(|u|^2).$$

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Conclusion. Passing to the limit as $\varepsilon \rightarrow 0$ in the integral equation satisfied by u_ε .

$$u_\varepsilon(t) = U(t)u_0 - i \int_0^t U(t-s)(F\varepsilon(u_\varepsilon(s)) + f) ds.$$

We deduce that

$$\int_0^t U(t-s)(F\varepsilon(u_\varepsilon(s)) + f) ds \rightarrow \int_0^t U(t-s)(F(u(s)) + f) ds.$$

Thus

$$u_\varepsilon(t) \rightarrow u(t).$$

This concludes the proof of Proposition 1.

References

- [1] M. Abounouh and O. Goubet, Attractor for damped cubic Schrödinger equation on a two dimensional thin domain, *Differential Integral Equations*, vol. 13, no. 1-3, pp. 311-340, 2000.
 - [2] H. Brezis, *Analyse fonctionnelle*, Édition Masson, Paris, 1983.
 - [3] T. Cazenave, An introduction to nonlinear Schrödinger equations, *Textos de Métodos Matemáticos*, 26, 1993.
 - [4] T. Cazenave and A. Haraux, *Introduction aux problèmes d'évolution semi-linéaires*, Ellipses, Paris, 1990.
 - [5] A. Dabaa, Comportement asymptotique des solutions d'un système d'équations de Schrödinger-Poisson sur un domaine borné de R^3 , *Annales Mathématiques Blaise Pascal*, vol. 17, no. 1, pp. 199–232, 2010.
 - [6] J. M. Ghidaglia, Finite dimensional behaviour for weakly damped driven Schrödinger equations, *Ann. Inst. Henri Poincaré*, vol. 5, pp. 365–405, 1988.
 - [7] O. Goubet, Regularité of the attracteurs for a weakly damped non linear Schrödinger equation, *Analysis*, vol. 60, pp. 99–119, 1995.
 - [8] P. Laurençot, Long time behaviour for weak damped driven non linear Schrödinger equation in R^n , $n \leq 3$, *NoDEA*, 1995.
 - [9] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation: Self-focusing and wave collapse*, Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999.
 - [10] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Appl. Math. Sc. 68, Springer-Verlag, New York, Second Edition, 1997.
 - [11] I. Białyński-Birula and J. Mycielski, Nonlinear wave mechanics, *Annals of Physics*, vol. 100, no. 1-2, pp. 62–93, 1976.
 - [12] R. Carles and I. Gallagher, Universal dynamics for the logarithmic Schrödinger equation, *Duke Mathematical Journal*, vol. 167, no. 16, pp. 3105–3141, 2018.
 - [13] T. Cazenave, Stable solutions of the logarithmic Schrödinger equation, *Nonlinear Analysis: Theory, Methods & Applications*, vol. 7, no. 10, pp. 1127–1140, 1983.
 - [14] H. Hajaiej, On the regularized logarithmic Schrödinger equation, *Communications on Pure and Applied Analysis*, vol. 12, no. 5, pp. 2003–2015, 2013.
 - [15] C. Ji and V. Rădulescu, Multi-peak solutions for logarithmic Schrödinger equations, *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 149, no. 6, pp. 1453–1468, 2019.
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