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A Dynamic Center–Spread Framework for Uncertainty Propagation in Linear Fuzzy Differential Systems

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Abstract
This paper proposes a dynamic center–spread framework for uncertainty propagation in linear fuzzy differential systems under strongly generalized Hukuhara (SGH) differentiability. The key novelty lies in treating the uncertainty magnitude $s(t)$ as an independent dynamic variable, rather than deriving it indirectly from membership functions or α-level sets. The uncertain state is represented as $\tilde{y}(t) = G(c(t), s(t))$, where $c(t)$ denotes the nominal trajectory and $s(t)$ governs uncertainty evolution. The original fuzzy problem is decomposed into two independent deterministic equations: a center equation and a spread equation. Under standard continuity and Lipschitz conditions, existence and uniqueness of admissible solutions are established. The framework also introduces a center–spread Laplace representation by applying the classical Laplace transform separately to each component, and preserves consistency with the deterministic model when $s(t) = 0$. Analytical examples and numerical experiments demonstrate the framework's ability to describe uncertainty attenuation and amplification independently of the nominal system behavior.

Keywords: Fuzzy differential equations; center–spread uncertainty; strongly generalized Hukuhara differentiability; uncertainty propagation; fuzzy Laplace representation.

إطار ديناميكي للمركز-الانتشار لنشر عدم اليقين في الأنظمة التفاضلية الضبابية الخطية

الملخص:
يقترح هذا البحث إطاراً ديناميكياً للمركز-الانتشار لنشر عدم اليقين في الأنظمة التفاضلية الضبابية الخطية في إطار قابلية الاشتقاق المعممة من نوع هوكوهارا (SGH). يكمن الابتكار الأساسي في معاملة مقدار عدم اليقين $s(t)$ كمتغير ديناميكي مستقل، بدلاً من استنتاجه بصورة غير مباشرة من دوال العضوية أو مجموعات القطع - α، يتم تمثيل الحالة الضبابية بالصيغة $\tilde{y}(t) = G(c(t), s(t))$، حيث يمثل $c(t)$ المسار الاسمي ويمثل $s(t)$ تطور عدم اليقين. يتم تحليل المشكلة الضبابية الأصلية إلى معادلتين حتميتين مستقلتين: معادلة المركز ومعادلة الانتشار. تحت شروط الاستمرارية وليبتشيتز القياسية، يتم إثبات وجود ووحدة الحلول المقبولة. يقدم الإطار أيضاً تمثيل لابلاس للمركز-الانتشار من خلال تطبيق تحويل لابلاس الكلاسيكي على كل مكون على حده، ويحافظ على الاتساق مع النموذج الحتمي عندما يكون $s(t) = 0$. تظهر الأمثلة التحليلية والتجارب العددية قدرة الإطار على وصف تضاؤل وتضخم عدم اليقين بصورة مستقلة عن سلوك النظام الاسمي.

الكلمات المفتاحية: المعادلات التفاضلية الضبابية؛ عدم يقين المركز-الانتشار؛ قابلية الاشتقاق المعممة من نوع هوكوهارا؛ نشر عدم اليقين؛ تمثيل لابلاس الضبابي.

1 Introduction

Mathematical models frequently involve uncertain parameters, imprecise initial conditions, and incomplete information. Fuzzy set theory, introduced by Zadeh, provides a fundamental framework for representing such uncertainty through membership functions [11]. The theory of fuzzy differential equations (FDEs) extends classical differential equations by allowing the state variables and forcing terms to be fuzzy-valued functions. However, extending classical differentiation to fuzzy-valued functions is not straightforward because the Hukuhara difference may not exist for arbitrary fuzzy numbers. This limitation motivated the development of generalized concepts of fuzzy differentiability, particularly the strongly generalized Hukuhara (SGH) differentiability, which provides a broader framework for fuzzy differential equations [10,3,2]. The SGH framework has been successfully used to study fuzzy initial value problems, numerical approximation, and linear fuzzy differential systems. In particular, it allows different uncertainty behaviors, including solutions whose support may increase or decrease with time [9,3]. Several analytical methods have been proposed for solving fuzzy differential equations, including fuzzy Laplace transform techniques. These methods extend classical transform approaches to fuzzy-valued systems and provide analytical solutions for several classes of problems [8,1]. Most existing approaches describe the uncertain solution through membership functions or through the family of α-cuts: $[\tilde{y}(t)]_\alpha = [\underline{y}_\alpha(t), \overline{y}_\alpha(t)]$. Although this representation is mathematically general, the evolution of uncertainty magnitude is usually obtained indirectly from the lower and upper solutions.

1.1 Research Motivation

The main motivation of this work is the need for an explicit description of uncertainty propagation. In many fuzzy differential models, the nominal evolution of the system and the evolution of uncertainty are embedded in a single fuzzy trajectory. Consequently, it becomes difficult to separately analyze:

- the deterministic motion of the system;
- the attenuation or amplification of uncertainty.

To overcome this limitation, we introduce a dynamic representation where uncertainty itself becomes a state variable. The proposed framework is not intended to introduce a new class of fuzzy numbers. Instead, it provides a dynamic representation for a structured class of fuzzy-valued solutions whose uncertainty can be characterized by a center variable and a spread variable.

$$\boxed{\tilde{y}(t) = G(c(t), s(t))} \tag{1}$$

where $c(t)$ represents the nominal state and $s(t) \geq 0$ represents the uncertainty magnitude. As will be shown in Section 3, this representation leads to independent evolution equations for the center and spread components. The essential idea is to consider uncertainty itself as a dynamic quantity governed by an evolution law, rather than as a static feature extracted from membership levels.

1.2 Main Contributions

The main contributions of this paper are summarized as follows:

1. A dynamic center–spread framework is introduced for a class of Gaussian-type fuzzy differential systems.
2. The uncertain state is decomposed into independent center and spread components, allowing explicit analysis of uncertainty propagation.
3. An existence and uniqueness theorem is established for admissible center–spread solutions.
4. A center–spread Laplace representation is developed by applying the classical Laplace transform separately to the two uncertainty components.
5. The deterministic reduction property is proved when the spread component vanishes.
6. Analytical and numerical examples demonstrate uncertainty attenuation and uncertainty amplification, including second-order and time-varying systems.

Note 1: The framework is developed for Gaussian fuzzy numbers. The core idea of treating uncertainty as a dynamic state variable is general and can be adapted to other symmetric fuzzy number families, but such extensions are not considered in this work.

1.2.1 Clarification of Novelty

It is important to emphasize that the proposed framework does not merely consist of transforming a fuzzy differential equation into two classical ODEs—a technique already present in the literature (see, e.g., [7,6,2]). The novelty resides in the following fundamental aspects:

1. Uncertainty as a state variable: The spread variable $s(t)$ is introduced as an independent state variable with its own dynamics, rather than being derived a posteriori from the lower and upper solutions.
2. Direct interpretation of uncertainty dynamics: The framework provides a direct interpretation of uncertainty attenuation ($s'(t) < 0$) and amplification ($s'(t) > 0$) as intrinsic properties of the system, not as secondary effects.
3. Separate transform-domain analysis: The center–spread Laplace representation $\mathcal{L}_{cs}\{\tilde{y}(t)\} = G(C(p), S(p))$ enables independent frequency-domain analysis of nominal and uncertainty responses—a feature not present in standard fuzzy Laplace methods [8,1].

The remainder of the paper is organized as follows: Section 2 introduces the mathematical preliminaries and the center–spread representation. Section 3 formulates the uncertain differential problem. Section 4 establishes existence and uniqueness results. Section 5 presents the center–spread Laplace representation.

The classical reduction property and fundamental properties are discussed in Sections 6 and 7. Numerical experiments and comparisons are presented in the final sections.

2 Preliminaries and Mathematical Background

This section presents the mathematical concepts required for the proposed center–spread uncertainty framework. The definitions of fuzzy numbers, level sets, and strongly generalized Hukuhara differentiability are briefly reviewed.

2.1 Fuzzy Numbers and Alpha-Cuts

Let $\mathbb{R}_{\mathcal{F}}$ denote the set of all fuzzy numbers defined on the real line. A fuzzy number is a mapping $\tilde{A}: \mathbb{R} \to [0,1]$ satisfying the following properties:

- normality;
- convexity;
- upper semi-continuity;
- compact support.

The concept of fuzzy sets was introduced by Zadeh as a general framework for representing imprecise information [11]. For each membership level $\alpha \in [0,1]$, the corresponding α-cut is defined as $[\tilde{A}]_\alpha = \{x \in \mathbb{R}: \tilde{A}(x) \geq \alpha\}$. For a fuzzy number, the α-cut is a closed interval: $[\tilde{A}]_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha]$. The level-set representation provides a powerful tool for transforming fuzzy differential equations into families of interval or deterministic problems [7].

2.2 Strongly Generalized Hukuhara Differentiability

The classical Hukuhara difference is defined only under restrictive conditions, which may prevent the existence of fuzzy derivatives for general fuzzy-valued functions. To overcome this limitation, generalized forms of Hukuhara differentiability were introduced [10,2].

Definition 1. Let $\tilde{y}: (a,b) \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function. The function $\tilde{y}(t)$ is called strongly generalized Hukuhara differentiable at t_0 if there exists $\tilde{y}'(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that the generalized Hukuhara difference limits exist. The SGH derivative allows different differentiability types depending on the behavior of the fuzzy support. This flexibility makes it possible to describe both increasing and decreasing uncertainty widths [3].

2.3 Parametric Representation of Fuzzy Functions

A fuzzy-valued function can be described through its level functions:

$$\boxed{\tilde{y}(t) = \left[\underline{y}(t,\alpha), \overline{y}(t,\alpha)\right], \alpha \in [0,1].} \tag{2}$$

The lower and upper functions satisfy $\underline{y}(t,\alpha) \leq \overline{y}(t,\alpha)$. Traditional approaches obtain the fuzzy solution by solving the evolution of these two boundary functions.

2.4 Center–Spread Representation

In this work, we adopt a Gaussian-type uncertainty profile as canonical representation. The membership function is defined as:

$$\boxed{\mu_{\tilde{y}}(x,t) = \exp\left(-\frac{(x-c(t))^2}{2s^2(t)}\right), s(t) > 0.} \tag{3}$$

Scope clarification: The choice of Gaussian profile is motivated by its smoothness, infinite support, and the fact that uncertainty is fully characterized by a single spread parameter $s(t)$. The theoretical framework can be extended to other symmetric profiles (triangular, trapezoidal, etc.) by modifying the reconstruction operator G, but such extensions are outside the scope of this work. The Gaussian case serves as a proof of concept for the center-spread dynamic approach.

Remark 1. The center–spread representation is a structured representation of fuzzy-valued states and does not replace the general theory of fuzzy numbers. It is applicable to fuzzy quantities for which the uncertainty can be described through a center and a spread parameter.

2.5 Alpha-Cut of the Center–Spread Model

The α-cut is obtained from $\mu_{\tilde{y}}(x,t) \geq \alpha$.

Therefore, $\exp\left(-\frac{(x-c(t))^2}{2s^2(t)}\right) \geq \alpha$.
Taking logarithms gives $(x - c(t))^2 \leq -2s^2(t)\ln\alpha$.
Hence,

$$\boxed{[\tilde{y}(t)]_\alpha = \left[c(t) - s(t)\sqrt{-2\ln\alpha},\, c(t) + s(t)\sqrt{-2\ln\alpha}\right].} \tag{4}$$

This formula establishes the relationship between the proposed dynamic variables and the classical level-set representation.

Remark 2. The center–spread representation is not intended to replace the general theory of fuzzy numbers. It defines a structured subclass of fuzzy-valued states for which the uncertainty magnitude has an explicit dynamic interpretation.

2.5.1 Relation with Alpha-Level Representation

The disappearance of uncertainty must be distinguished from selecting the maximum membership level.
For the center–spread representation:
At $\alpha = 1$ into (4), we obtain $[\tilde{y}(t)]_1 = [c(t), c(t)]$.
This represents the core of the fuzzy number only.
However, uncertainty still exists whenever $s(t) > 0$.
Therefore: $\alpha = 1 \nRightarrow$ deterministic solution.
The deterministic reduction requires: $s(t) = 0$.

2.5.2 Numerical Visualization

The following figures illustrate the proposed approach:

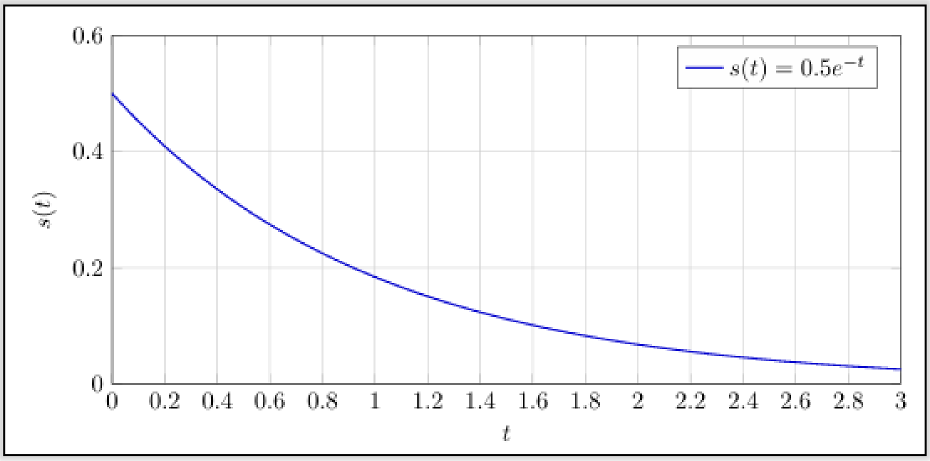

Figure 1: Decay of uncertainty spread.

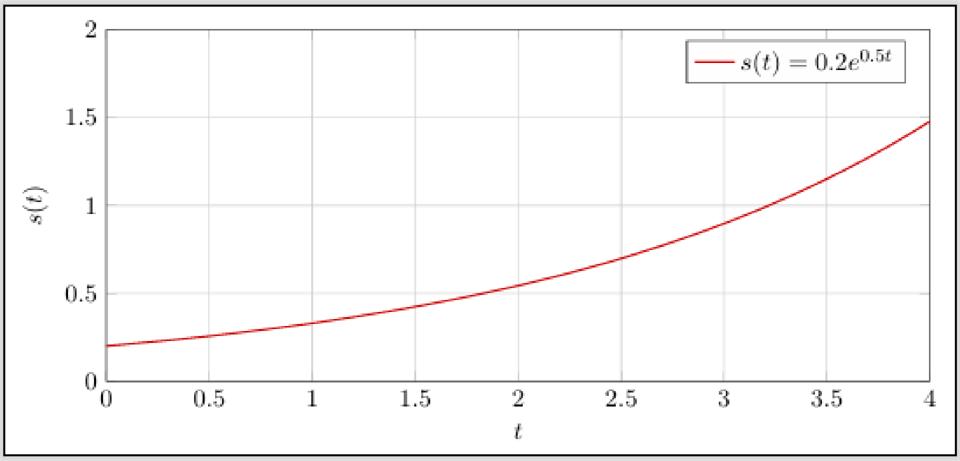

Figure 2: Growth of uncertainty spread.

2.5.3 Uncertainty Width Dynamics

The width of the interval is therefore $W_\alpha(t) = 2s(t)\sqrt{-2\ln\alpha}$. This equation shows explicitly that the uncertainty width is controlled only by the spread dynamics.

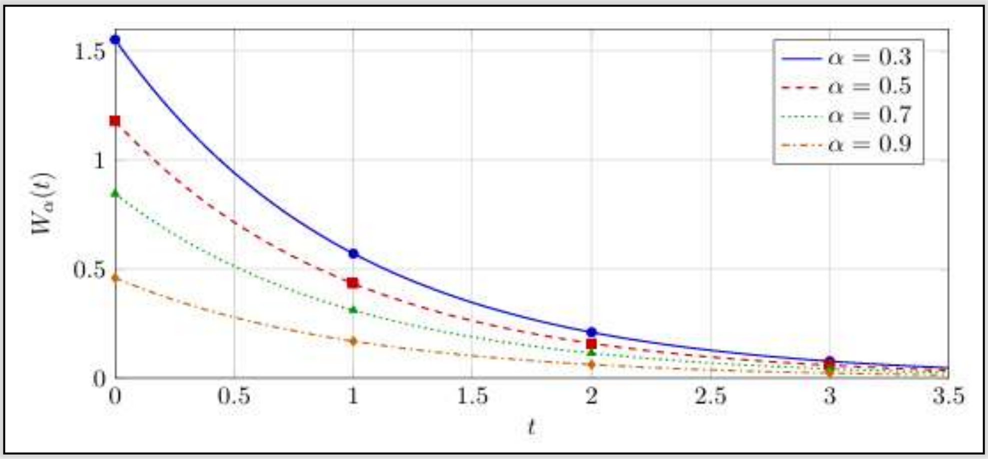

Figure 3: Evolution of the uncertainty width $W_\alpha(t)$ for different membership levels α.

2.5.4 Rate of Width Change

The derivative of the width is: $W_\alpha'(t) = 2s'(t)\sqrt{-2\ln\alpha}$. This shows that the uncertainty width evolves according to the sign of $s'(t)$. The interpretation of this result is discussed in Section 3.6.

2.6 Extraction of Initial Center and Spread from Fuzzy Data

Given a Gaussian fuzzy number $\tilde{Y}$ with membership function:

$$\mu_{\tilde{Y}}(x,t) = \exp\left(-\frac{(x-c_0)^2}{2{s_0}^2}\right),$$

the parameters (c_0, s_0) are extracted as follows:

Algorithm 1: Extraction for Gaussian Fuzzy Numbers

1. Compute the α-cuts: $[\tilde{Y}]\alpha = [\underline{Y}_\alpha, \overline{Y}_\alpha], \alpha \in [0,1]$
2. Extract the center: $c_0 = \frac{\underline{Y}_0 + \overline{Y}_0}{2}$ (Alternatively, $c_0 = argmax_x \mu\tilde{Y}(x)$)
3. Extract the spread: For any $\alpha \in (0,1)$, $s_0 = \frac{\overline{Y}_\alpha - \underline{Y}_\alpha}{2\sqrt{-2ln\alpha}}$

Algorithm 2: Extraction for Higher-Order Derivatives

For the k-th derivative initial condition $D^k_{SGH}\tilde{y}(0) = \tilde{Y}_k$:

1. Compute the α-cuts of $\tilde{Y}_k$: $[\underline{Y}_{k,\alpha}, \overline{Y}_{k,\alpha}]$
2. Extract the center: $c_k = \frac{\underline{Y}_{k,0} + \overline{Y}_{k,0}}{2}$
3. Extract the spread: $s_k = \frac{\overline{Y}_{k,\alpha} - \underline{Y}_{k,\alpha}}{2\sqrt{-2ln\alpha}}$

Algorithm 3: Extraction from Gaussian Membership Data

Given sampled data from the membership function $\mu(x_i)$:

1. Fit a Gaussian function $\mu(x) = \exp\left(-\frac{(x-c_0)^2}{2{s_0}^2}\right)$ using nonlinear least squares.
2. Alternatively, use the maximum likelihood estimates: $c_0 = \frac{\sum_i x_i \mu(x_i)}{\sum_i \mu(x_i)}$ and ${s_0}^2 = \frac{\sum_i (x_i - c_0)^2 \mu(x_i)}{\sum_i \mu(x_i)}$

Example 0: Given the Gaussian fuzzy number with α-cuts: $[\tilde{y}(0)]_\alpha = [1 - 0.5\sqrt{-2ln\alpha}, 1 + 0.5\sqrt{-2ln\alpha}]$.
We extract:

- $c_0 = 1$
- $s_0 = 0.5$

For the first derivative initial condition: $[\tilde{y}'(0)]_\alpha = [0 - 0.2\sqrt{-2ln\alpha}, 0 + 0.2\sqrt{-2ln\alpha}]$

We extract:
- $c_0 = 0$
- $s_0 = 0.2$

Table 1: Extraction Summary

Initial Condition	α-cuts	c_k	s_k
$\tilde{y}(0)$	$[1 - 0.5\sqrt{-2ln\alpha}, 1 + 0.5\sqrt{-2ln\alpha}]$	1.0	0.5
$\tilde{y}'(0)$	$[0 - 0.2\sqrt{-2ln\alpha}, 0 + 0.2\sqrt{-2ln\alpha}]$	0.0	0.2

3 Dynamic Center–Spread Formulation of the Fuzzy Differential System

This section introduces the proposed dynamic center–spread formulation for a class of linear fuzzy differential systems. The main objective is to separate the nominal evolution of the system from the propagation of uncertainty.

3.1 General Linear Fuzzy Initial Value Problem

Consider the following n-th order linear fuzzy differential equation:

$$\boxed{D_{SGH}^{n}\tilde{y}(t) + \sum_{k=0}^{n-1} a_k(t) D_{SGH}^{k}\tilde{y}(t) = \tilde{f}(t),} \tag{5}$$

with fuzzy initial conditions $D_{SGH}^{k}\tilde{y}(0) = \tilde{Y}_k,\ k = 0,1,\dots,n-1$.

Here, $\tilde{y}(t) \in \mathbb{R}_{\mathcal{F}}$ is the uncertain state, $\tilde{f}(t) \in \mathbb{R}_{\mathcal{F}}$ is the uncertain input, and $a_k(t)$ are continuous real-valued coefficients. The derivative D_{SGH} denotes the strongly generalized Hukuhara derivative, which allows a larger class of fuzzy-valued functions compared with the classical Hukuhara derivative [3,2].

3.2 Dynamic Center–Spread Decomposition

The proposed framework assumes that the fuzzy state can be represented as (1). Similarly, the fuzzy forcing term is represented as

$$\boxed{\tilde{f}(t) = G(f_c(t), f_s(t)).} \tag{6}$$

The initial fuzzy conditions are decomposed as $\tilde{Y}_k = G(c_k, s_k),\ k = 0,1,\dots,n-1$. The admissibility condition requires $s(t) \geq 0$. This condition guarantees that the reconstructed fuzzy state remains valid.

The main structure of the proposed framework is illustrated in Figure 4, which conceptual representation of the proposed dynamic center–spread framework. The fuzzy differential system is decomposed into nominal center dynamics and uncertainty dynamics, followed by reconstruction of the fuzzy-valued solution.

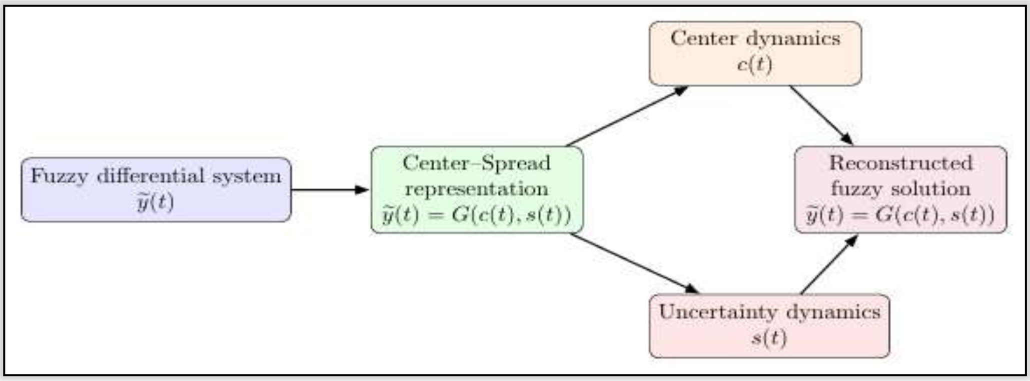

Figure 4: Flowchart

3.3 Derivation of the Center Equation

Substituting the center–spread representation into Equation (5) and separating the deterministic component yields the center equation:

$$\boxed{c^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t) c^{(k)}(t) = f_c(t).} \tag{7}$$

The corresponding initial conditions are $c^{(k)}(0) = c_k,\ k = 0,1,\dots,n-1$.

Equation (7) represents the evolution of the nominal trajectory without uncertainty.

3.4 Derivation of the Spread Equation

The uncertainty component is described by the spread variable $s(t)$. Its evolution is governed by an independent dynamic equation: $\mathcal{D}_s(s(t)) = f_s(t)$, where $\mathcal{D}_s$denotes the spread evolution operator induced by the chosen fuzzy differentiability framework. For the linear case, the spread equation takes the form

$$\boxed{s^{(n)}(t) + \sum_{k=0}^{n-1} a_k\,(t) s^{(k)}(t) = f_s(t).} \tag{8}$$

The initial spread conditions are $s^{(k)}(0) = s_k,\ k = 0,1, \ldots, n-1.$

3.5 Equivalent Center–Spread System

Combining (7) and (8), the original fuzzy problem is transformed into the coupled system (9)-(10), which is then solved as two independent deterministic equations. The fuzzy solution is reconstructed via (1).

$$\boxed{\begin{cases} c^{(n)}(t) + \sum_{k=0}^{n-1} a_k\,(t) c^{(k)}(t) = f_c(t), \\ s^{(n)}(t) + \sum_{k=0}^{n-1} a_k\,(t) s^{(k)}(t) = f_s(t), \end{cases}} \tag{9}$$

with initial conditions:

$$\boxed{\begin{cases} c^{(k)}(0) = c_k, \\ s^{(k)}(0) = s_k, \end{cases} \qquad k = 0,1, \ldots, n-1.} \tag{10}$$

The linear system (9) exhibits structural symmetry between the center and spread equations. This symmetry is a feature of the linear case and reflects the following facts:

1. Same dynamics: Uncertainty propagates through the same linear operator $\mathcal{L} = \frac{d^m}{dt^m} + \sum_{k=0}^{n-1} a_k(t) \frac{d^k}{dt^k}$ as the nominal state. This is a natural consequence of the linearity of the original fuzzy system.
2. Separation of inputs: The center is driven by the deterministic input $f_c(t)$, while the spread is driven by the uncertainty input $f_s(t)$. This separation allows independent analysis of how the system and the uncertainty source affect the evolution of uncertainty.
3. Independence of initial conditions: The center and spread have independent initial conditions (c_k and s_k), allowing different initial behaviors.

The symmetry does not imply redundancy because:

- The center and spread typically have different initial conditions
- The forcing functions $f_c(t)$ and $f_s(t)$ are generally different
- The spread must satisfy the admissibility condition $s(t) \geq 0$
- The physical interpretation of $c(t)$ and $s(t)$ is fundamentally different.

Generalization: In the nonlinear case, the center and spread equations may have different structures:

$$\boxed{\begin{cases} c^{(n)}(t) + \sum_{k=0}^{n-1} a_k\,(t,c,s) c^{(k)}(t) = f_c(t,c.s), \\ s^{(n)}(t) + \sum_{k=0}^{n-1} a_k\,(t,c,s) s^{(k)}(t) = f_s(t,c,s), \end{cases}} \tag{11}$$

where $a_k \neq b_k$ in general. The linear case (9) is a special subclass where $a_k = b_k$.

3.6 Interpretation of the Model

The proposed formulation provides a direct interpretation of uncertainty evolution. The center equation determines the expected behavior of the system, whereas the spread equation determines how uncertainty changes with time. From Section 2.5.4, we have:

- If $s'(t) < 0$: the uncertainty width decreases (uncertainty attenuation).
- If $s'(t) > 0$: the uncertainty width increases (uncertainty amplification).
- If $s'(t) = 0$: the uncertainty width remains constant.

This interpretation distinguishes the proposed framework from approaches where uncertainty is obtained only indirectly from the lower and upper solutions. In classical α-cut methods, attenuation or amplification is inferred from the evolution of the lower and upper bounds, whereas in the proposed framework, it is explicitly represented by the spread variable $s(t)$, which acts as an independent state variable with its own dynamics. This allows for separate analysis of nominal behavior and uncertainty propagation.

4 Existence and Uniqueness of Admissible Center–Spread Solutions

This section establishes the existence and uniqueness of the proposed center–spread solution. The result is obtained by applying the classical existence and uniqueness theory for ordinary differential equations to the decomposed center and spread system.

4.1 Assumptions

Consider the center–spread system given by (9). We impose the following assumptions.

1. The coefficients $a_k(t),\ k = 0, \dots, n-1$ are continuous on $[0, T]$.
2. The forcing functions $f_c(t), f_s(t)$ are continuous on $[0, T]$.
3. The reconstruction operator $G(c, s)$ is continuous for all admissible values of s.
4. Invariance condition for spread admissibility: to guarantee that the spread variable remains non-negative, i.e., $s(t) \geq 0$ for all $t \in [0, T]$, we impose the following sufficient condition:

$$s^{(k)}(0) \geq 0, k = 0,1, \dots, n-1, \text{ and } f_s(t) \geq 0 \text{ for all } t \in [0, T].$$

Under these conditions, the solution of the linear spread equation remains non-negative by the comparison principle for ordinary differential equations (see Theorem 2 in Section 4.4).

Remark 3: These conditions are sufficient but not necessary. In cases where $f_s(t)$ may take negative values, the admissibility of the spread solution must be verified a posteriori. Alternatively, a logarithmic transformation $s(t) = e^{r(t)}$can be employed to guarantee $s(t) > 0$ automatically, though this introduces nonlinearity into the spread dynamics. Such extensions are beyond the scope of this linear framework and are left for future research.

4.2 First-Order System Representation

Define the state vectors $C(t) = (c(t), c'(t), \dots, c^{(n-1)}(t))^T$, and $S(t) = (s(t), s'(t), \dots, s^{(n-1)}(t))^T$. Then the center equation can be rewritten as the first-order system $C'(t) = F_c(t, C(t))$, where $F_c: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$. Similarly, the spread equation becomes $S'(t) = F_s(t, S(t))$. The complete system is therefore

$$\boxed{\begin{cases} C'(t) = F_c(t, C(t)), \\ S'(t) = F_s(t, S(t)). \end{cases}} \tag{12}$$

4.3 Existence and Uniqueness Theorem

Theorem 1. Assume that the functions $F_c(t, C)$ and $F_s(t, S)$ are continuous in t and locally Lipschitz with respect to their state variables. Assume further that the invariance conditions of Theorem 2 hold. Then, for any admissible initial conditions $C(0) = C_0,\ S(0) = S_0$, with $s_0 \geq 0$, there exists a unique solution pair $(C(t), S(t))$ on $[0, T]$. Consequently, the reconstructed uncertain function $\tilde{y}(t) = G(c(t), s(t))$ is the unique admissible center–spread solution.

Proof. The center and spread equations are written as the first-order system from (12). By assumption, both vector fields are continuous in time and locally Lipschitz with respect to the state variables. Therefore, according to the classical Picard–Lindelöf theorem, the initial value problems $C'(t) = F_c\big(t, C(t)\big),\ C(0) = C_0$, and $S'(t) = F_s\big(t, S(t)\big),\ S(0) = S_0$, possess unique solutions. Hence, there exists a unique pair $(C(t), S(t))$ satisfying the decomposed system. By Theorem 2, the spread component satisfies $s(t) \geq 0$, for all $t \in [0, T]$, ensuring that the reconstruction operator remains well-defined. Since $G(c, s)$ is continuous, the function $\tilde{y}(t) = G(c(t), s(t))$ is uniquely determined. Therefore, the proposed center–spread solution exists and is unique. □

Remark 4: Explicit Solution Formula

For the linear system (9), the solutions can be expressed explicitly using the state transition matrix:

$$C(t) = \Phi(t, 0)C_0 + \int_0^t \Phi(t, \tau)F_c(\tau)d\tau,\ S(t) = \Phi(t, 0)S_0 + \int_0^t \Phi(t, \tau)F_s(\tau)d\tau,$$

where $\Phi(t, \tau)$ satisfies $\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau)$, $\Phi(\tau, \tau) = I$.

Corollary 1: Boundedness and Stability

If all eigen-values of $A(t)$ have negative real parts uniformly in t, then the solutions $C(t)$ and $S(t)$ are uniformly bounded on $[0, \infty)$. Consequently, $s(t)$ decays exponentially, corresponding to uncertainty attenuation.

Remark 5: Comparison with Classical FDE Theory

Unlike general fuzzy differential equations, where existence and uniqueness require stronger assumptions on the fuzzy right-hand side (see [7,3,2]), the center-spread framework reduces the problem to classical ODEs. This simplification is a key advantage of the proposed approach, as it allows explicit solution formulas and direct application of classical stability theory.

4.4 Invariance of Spread Admissibility

The following result establishes the preservation of the non-negativity condition for the spread variable, which is essential for the validity of the reconstructed fuzzy solution.

Theorem 2: Invariance of Spread Admissibility

Consider the spread equation (8) with initial conditions $s^{(k)}(0) = s_k, k = 0, \dots, n-1$. Assume:

1. $s_k \geq 0$ for all $k = 0, \dots, n-1$,
2. $f_s(t) \geq 0$ for all $t \in [0, T]$,
3. The coefficients $a_k(t)$ are continuous on $[0, T]$.

Then the solution satisfies $s(t) \geq 0$for all $t \in [0, T]$.

Proof. For the first-order case ($n = 1$), the solution of (8) is given explicitly by:

$$s(t) = s_0 e^{-\int_0^t a_0(\tau)d\tau} + \int_0^t f_s(\tau) e^{-\int_\tau^t a_0(\xi)d\xi} d\tau.$$

Since $s_0 \geq 0$, $f_s(\tau) \geq 0$, and the exponential terms are strictly positive for all $\tau \in [0, T]$, both terms on the right-hand side are non-negative. Hence $s(t) \geq 0$ for all $t \in [0, T]$. For the general n-th order case ($n > 1$), the proof follows from the comparison principle for higher-order linear differential equations. Let $\underline{s}(t) \equiv 0$ be the lower solution. Since $\underline{s}^{(k)}(0) = 0 \leq s_k$ for all $k = 0, \dots, n-1$, and $\underline{s}^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t) \underline{s}^{(k)}(t) = 0 \leq f_s(t)$, the comparison principle ensures that $s(t) \geq \underline{s}(t) = 0$ for all $t \in [0, T]$. Therefore, $s(t) \geq 0$. □

Remark 6: Theorem 2 provides a sufficient condition for the admissibility of the spread solution. If $f_s(t)$ is not non-negative, the invariance property is not guaranteed and must be verified on a case-by-case basis for each specific problem. Extensions to general forcing functions can be addressed through nonlinear transformations such as $s(t) = e^{r(t)}$, which are beyond the scope of this linear framework and are left for future research.

5 Center–Spread Laplace Representation

The Laplace transform is an important analytical tool for solving linear differential equations. In fuzzy differential equations, several extensions of the Laplace transform have been proposed to obtain analytical solutions of fuzzy-valued problems [8,1]. In this section, a center–spread Laplace representation is introduced by applying the classical Laplace transform independently to the center and spread components.

5.1 Laplace Transform of the Center Component

Let the center function be $c(t)$, and assume that it is of exponential order. The Laplace transform of the center component is defined by

$$\boxed{C(p) = \mathcal{L}\{c(t)\} = \int_0^\infty e^{-pt} c(t)\, dt.} \tag{13}$$

For the first derivative, $\mathcal{L}\{c'(t)\} = pC(p) - c(0)$.

Similarly,

$$\boxed{\mathcal{L}\{c^{(n)}(t)\} = p^n C(p) - \sum_{j=0}^{n-1} p^{n-1-j} c^{(j)}(0).} \tag{14}$$

Therefore, the transformed center equation can be obtained directly from the classical Laplace rules.

5.2 Laplace Transform of the Spread Component

The spread function $s(t) \geq 0$ is treated as an independent uncertainty state. Its Laplace transform is defined as

$$\boxed{S(p) = \mathcal{L}\{s(t)\} = \int_0^\infty e^{-pt} s(t)\, dt.} \tag{15}$$

The derivative property gives

$$\mathcal{L}\{s'(t)\} = pS(p) - s(0).$$

More generally,

$$\boxed{\mathcal{L}\{s^{(n)}(t)\} = p^n S(p) - \sum_{j=0}^{n-1} p^{n-1-j}\, s^{(j)}(0).} \tag{16}$$

5.3 Definition of the Center–Spread Laplace Representation

Definition 2. For a center–spread fuzzy-valued function $\tilde{y}(t) = G(c(t), s(t))$, the center–spread Laplace representation is defined as

$$\boxed{\mathcal{L}_{cs}\{\tilde{y}(t)\} = G(C(p), S(p)).} \tag{17}$$

The operator $\mathcal{L}_{cs}$ is not considered a new integral transform. It represents the application of the classical Laplace transform to the independent components of the uncertainty model.

5.4 Transformed Center–Spread System

Applying the Laplace transform to the center equation gives

$$\boxed{(p^n + \sum_{k=0}^{n-1} a_k\, p^k)C(p) = F_c(p) + P_c(p),} \tag{18}$$

Where $P_c(p)$ contains the initial condition terms. Similarly, the spread equation becomes

$$\boxed{(p^n + \sum_{k=0}^{n-1} a_k\, p^k)S(p) = F_s(p) + P_s(p).} \tag{19}$$

Hence, the uncertainty problem is represented in the transform domain by two independent algebraic equations.

5.5 Consistency with the Classical Laplace Transform

A fundamental requirement of the proposed representation is the recovery of the classical case.

Proposition 1. If $s(t) = 0$, then $S(p) = 0$, and the center–spread Laplace representation reduces to the classical Laplace transform of the deterministic solution.

Proof. If $s(t) = 0$, then by definition $S(p) = \mathcal{L}\{0\} = 0$. Therefore, $\mathcal{L}_{cs}\{\tilde{y}(t)\} = G(C(p), 0)$. Since zero spread corresponds to the deterministic state, $\tilde{y}(t) = c(t)$, we obtain $\mathcal{L}_{cs}\{\tilde{y}(t)\} = C(p)$. Hence the proposed representation is consistent with the classical Laplace transform. □

6 Classical Reduction and Fundamental Properties

This section investigates the fundamental properties of the proposed center–spread framework. In particular, we prove the reduction to the classical deterministic system and establish several mathematical properties of the uncertainty evolution.

6.1 Classical Reduction Property

The proposed representation should contain the classical differential equation as a special case when the uncertainty completely disappears.

Theorem 3. Let $\tilde{y}(t) = G(c(t), s(t))$ be an admissible center–spread solution. If $s(t) = 0,\ t \in [0, T]$, then the fuzzy differential system reduces to the corresponding classical deterministic differential equation

$$c^{(n)}(t) + \sum_{k=0}^{n-1} a_k\,(t) c^{(k)}(t) = f_c(t).$$

Proof. When $s(t) = 0$, the reconstruction operator becomes $\tilde{y}(t) = G(c(t), 0)$. A zero spread represents a degenerate fuzzy number with no uncertainty. Therefore, $\tilde{y}(t) = c(t)$. Substituting into the original fuzzy differential equation eliminates the uncertainty component and leaves only the center equation:

$$c^{(n)}(t) + \sum_{k=0}^{n-1} a_k\,(t) c^{(k)}(t) = f_c(t).$$

Hence, the deterministic classical model is recovered. □

6.2 Continuity of the Uncertain Solution

Theorem 4. Assume that $c(t), s(t) \in C([0, T])$ and that the reconstruction operator G is continuous. Then $\tilde{y}(t) = G(c(t), s(t))$ is continuous on $[0, T]$.

Proof. Let $t_m \to t$. Because $c(t)$ and $s(t)$ are continuous, $c(t_m) \to c(t)$, and $s(t_m) \to s(t)$. The continuity of G implies $G(c(t_m), s(t_m)) \to G(c(t), s(t))$. Hence, $\tilde{y}(t)$ is continuous. □

6.3 Preservation of Spread Admissibility

The spread variable must preserve the condition $s(t) \geq 0$ to maintain a valid uncertainty representation.

Theorem 5. Assume that $s(0) \geq 0$ and that the spread equation satisfies the invariance condition of the non-negative domain. Then $s(t) \geq 0,\ t \in [0, T]$.

Proof. Consider the admissible set $\Omega_s = [0, \infty)$. By assumption, the spread vector field does not allow trajectories starting in Ω_s to leave this set. Since $s(0) \in \Omega_s$, the solution remains in Ω_s for all time. Therefore, $s(t) \geq 0$. □

6.4 Continuous Dependence on Initial Conditions

Theorem 6. Let $(c_1(t), s_1(t))$ and $(c_2(t), s_2(t))$ be two center–spread solutions corresponding to different initial conditions. If the center and spread systems satisfy Lipschitz conditions, then there exists a constant $K > 0$ such that

$$\boxed{|c_1(t) - c_2(t)| + |s_1(t) - s_2(t)| \leq K(|c_1(0) - c_2(0)| + |s_1(0) - s_2(0)|)} \tag{20}$$

Proof. The result follows from the continuous dependence theorem for ordinary differential equations applied to the equivalent first-order center–spread system. Since both components satisfy Lipschitz conditions, perturbations in the initial data remain bounded. The continuity of G transfers this property to the reconstructed fuzzy solution. □

6.5 Summary

The above results establish that the proposed framework satisfies the following requirements:

- Consistency with the classical deterministic model;
- Continuity of the uncertain trajectory;
- Preservation of admissible uncertainty;
- Stability with respect to initial uncertainty.

7 Analytical Examples and Numerical Experiments

This section presents analytical and numerical examples to demonstrate the behavior of the proposed center–spread framework. The examples illustrate how the nominal evolution and uncertainty propagation can be analyzed independently.

Example 1: Linear System with Uncertainty Attenuation

Problem: $D_{SGH}\tilde{y}(t) = -\lambda\tilde{y}(t), \quad \tilde{y}(0) = G(c_0, s_0)$

Solution: Using the proposed representation, $\tilde{y}(t) = G(c(t), s(t))$, the system is decomposed into

$$\begin{cases} c'(t) = -\lambda c(t), \\ s'(t) = -\lambda s(t). \end{cases}$$

Numerical values: $c_0 = 2,\ s_0 = 0.5,\ \lambda = 1$.

The analytical solutions are $c(t) = 2e^{-t}$, and $s(t) = 0.5e^{-t}$.

Therefore, the fuzzy solution is reconstructed as $\tilde{y}(t) = G(2e^{-t}, 0.5e^{-t})$.

The spread function satisfies $\lim_{t\to\infty} s(t) = 0$, which indicates that the uncertainty gradually disappears.

Alpha-cut ($\alpha = 0.5$):

$$[\tilde{y}(t)]_{0.5} = [1.4113e^{-t}, 2.5887e^{-t}]$$

To validate the proposed approach, we compare the results with the classical α-cut method for Example 1. Table 2 shows perfect agreement between the two methods for the Gaussian uncertainty profile.

Table 2: Comparison of proposed method with classical α-cut solution for Ex. 1 ($\alpha = 0.5$)

t	$c(t)$	$s(t)$	$\underline{y}_{0.5}(t)$ (ours)	$\underline{y}_{0.5}(t)$ (classical)
0.0	2.0000	0.5000	1.6674	1.6674
0.5	1.2131	0.3033	1.0110	1.0109
1.0	0.7358	0.1839	0.6132	0.6132
2.0	0.2707	0.0677	0.2257	0.2256
3.0	0.0996	0.0249	0.0830	0.0830

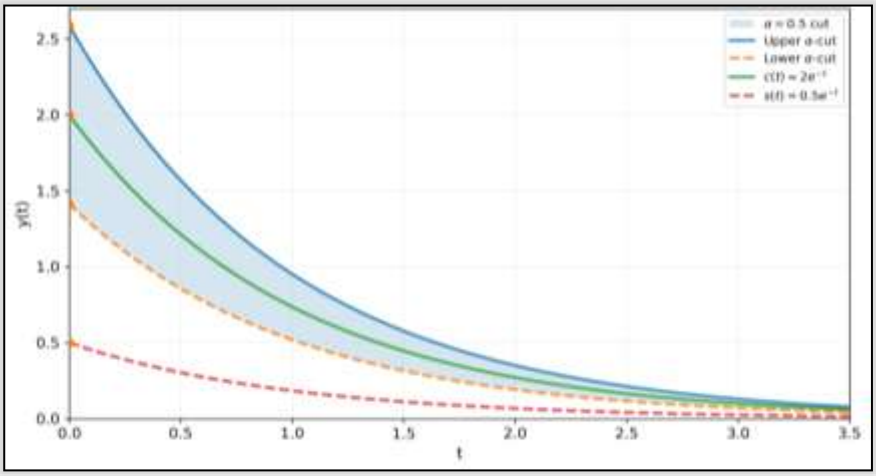

Figure 5: Linear System with Uncertainty Attenuation ($c_0 = 2,\ s_0 = 0.5,\ \lambda = 1$).

Figure 5 shows that both the nominal state and the uncertainty magnitude decay with time. In particular, $s(t) \to 0$, indicating progressive uncertainty attenuation.

Example 2: Linear System with Uncertainty Amplification
Problem: $D_{SGH}\tilde{y}(t) = \lambda\tilde{y}(t), \quad \tilde{y}(0) = G(c_0, s_0)$.
Solution: center–spread equations become $c'(t) = \lambda c(t)$, and $s'(t) = \lambda s(t)$.
Numerical values: $c_0 = 1,\ s_0 = 0.2,\ \lambda = 0.5$. Then $c(t) = e^{0.5t}$, and $s(t) = 0.2e^{0.5t}$.
Hence, $\tilde{y}(t) = G(e^{0.5t}, 0.2e^{0.5t})$. In this case, $\lim_{t\to\infty} s(t) = \infty$, which represents uncertainty amplification.
Alpha-cut ($\alpha = 0.5$): $[\tilde{y}(t)]_{0.5} = [0.7645e^{0.5t}, 1.2355e^{0.5t}]$

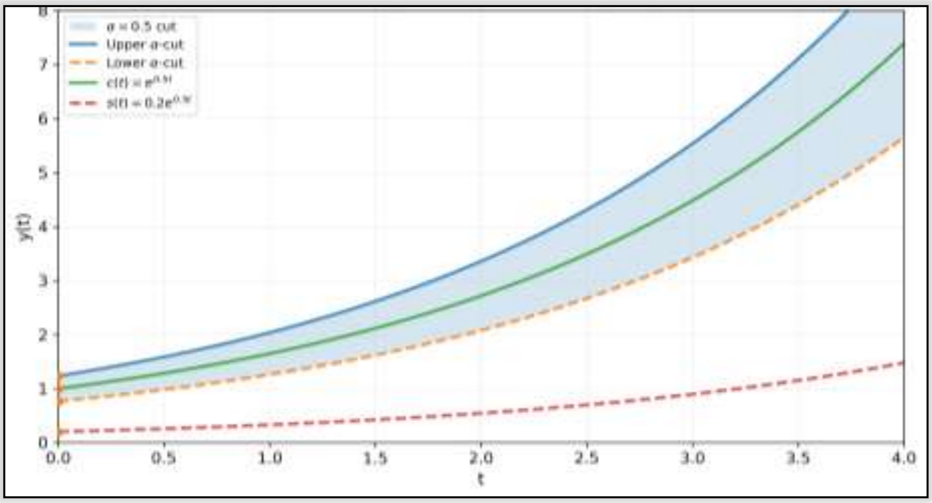

Figure 6: Linear System with Uncertainty Amplification ($c_0 = 1,\ s_0 = 0.2,\ \lambda = 0.5$).

Example 3: Triangular Fuzzy Initial Condition
Problem: $\tilde{y}(0) = (0.5, 1.5, 2.5), \quad D_{SGH}\tilde{y}(t) = -\tilde{y}(t)$
Extraction of parameters: The center and spread parameters are obtained as

$$c(0) = \frac{0.5+2.5}{2} = 1.5, \text{ and } s(0) = \frac{2.5-0.5}{2} = 1.$$

Solution: Consider the attenuation model $D_{SGH}\tilde{y}(t) = -\tilde{y}(t)$. According to the proposed decomposition,

$$\begin{cases} c'(t) = -c(t), \\ s'(t) = -s(t). \end{cases}$$

Therefore, $c(t) = 1.5e^{-t}$, and $s(t) = e^{-t}$.
The fuzzy solution is reconstructed as $\tilde{y}(t) = G(1.5e^{-t}, e^{-t})$.
For the membership level $\alpha = 0.5$, the uncertainty interval is

$$[\tilde{y}(t)]_{0.5} = [1.5e^{-t} - 0.8326e^{-t}, 1.5e^{-t} + 0.8326e^{-t}] = [0.6674e^{-t}, 2.3326e^{-t}].$$

Table 3: Numerical evolution of the center and spread

t	$c(t)$	$s(t)$	$[\tilde{y}(t)]_{0.5}$
0	1.5000	1.0000	[0.6674,2.3326]
0.5	0.9098	0.6065	[0.4049,1.4147]
1	0.5518	0.3679	[0.2454,0.8582]
2	0.2030	0.1353	[0.0904,0.3156]

The results confirm that the proposed framework preserves the fuzzy uncertainty structure while providing an explicit description of its temporal evolution through the spread function. Evolution of the $\alpha = 0.5$ cut of the fuzzy solution corresponding to the triangular initial condition $\tilde{y}(0) = (0.5, 1.5, 2.5)$. The center and the uncertainty boundaries contract toward zero as t increases.

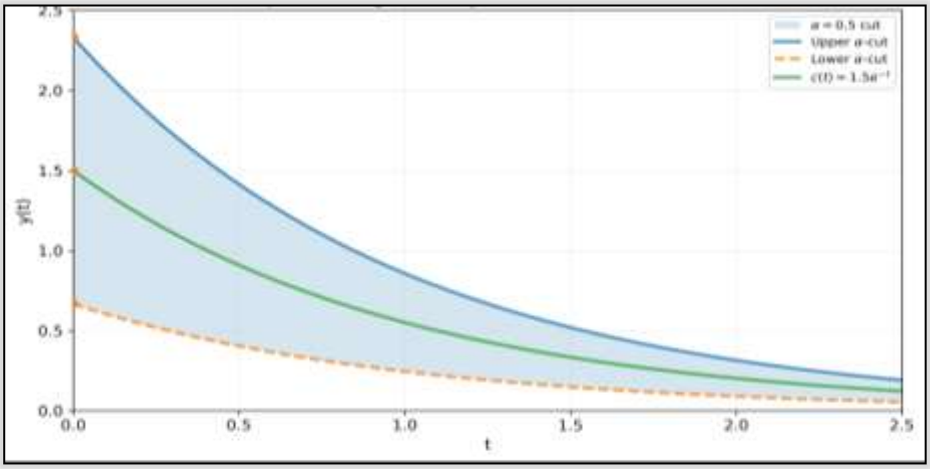

Figure 7: Triangular fuzzy initial condition $\tilde{y}(0) = (0.5,1.5,2.5)$.

Example 4: Second-Order System (Harmonic Oscillator)
Problem: $D_{SGH}^2\tilde{y}(t) + 2\zeta\omega_n D_{SGH}\tilde{y}(t) + \omega_n^2\tilde{y}(t) = 0$
Decomposition: $\begin{cases} c''(t) + 2\zeta\omega_n c'(t) + \omega_n^2 c(t) = 0, \\ s''(t) + 2\zeta\omega_n s'(t) + \omega_n^2 s(t) = 0, \end{cases}$
with initial conditions $c(0) = c_0,\ c'(0) = c_1,\ s(0) = s_0,\ s'(0) = s_1$.
Case A: Underdamped System ($\zeta < 1$)
The solutions are

$$c(t) = e^{-\zeta\omega_n t}\left(c_0\cos(\omega_d t) + \frac{c_1+\zeta\omega_n c_0}{\omega_d}\sin(\omega_d t)\right), \text{ and } s(t) = e^{-\zeta\omega_n t}\left(s_0\cos(\omega_d t) + \frac{s_1+\zeta\omega_n s_0}{\omega_d}\sin(\omega_d t)\right),$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$.
The uncertainty width evolves as $W_\alpha(t) = 2s(t)\sqrt{-2\ln\alpha}$.
Numerical values: $\zeta = 0.3,\ \omega_n = 2,\ c_0 = 1,\ c_1 = 0,\ s_0 = 0.3,\ s_1 = 0$

$$\omega_d = 2\sqrt{0.91} \approx 1.9079$$

$$c(t) = e^{-0.6t}(\cos(1.9079t) + 0.3145\sin(1.9079t))$$

$$s(t) = 0.3e^{-0.6t}(\cos(1.9079t) + 0.3145\sin(1.9079t))$$

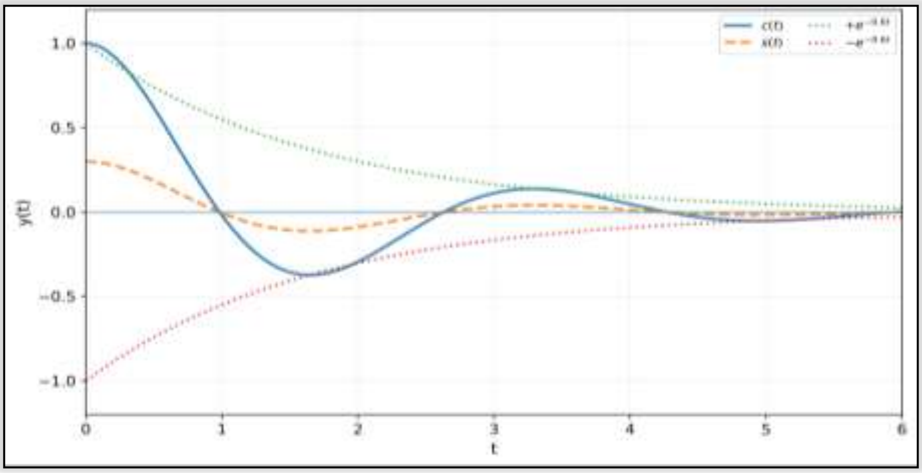

Figure 8A: Underdamped oscillator with $\zeta = 0.3,\ \omega_n = 2$.

Remark 7. Both the nominal trajectory and uncertainty amplitude decay with the same envelope $e^{-\zeta\omega_n t}$. However, the phase of uncertainty oscillations may differ from the nominal phase if the initial conditions differ.

Case B: Overdamped System ($\zeta > 1$)

The solutions are

$$c(t) = e^{-\zeta\omega_n t}\left(A_c e^{\omega_n\sqrt{\zeta^2-1}t} + B_c e^{-\omega_n\sqrt{\zeta^2-1}t}\right),$$
$$s(t) = e^{-\zeta\omega_n t}\left(A_s e^{\omega_n\sqrt{\zeta^2-1}t} + B_s e^{-\omega_n\sqrt{\zeta^2-1}t}\right),$$

where the coefficients are determined by initial conditions.

Numerical values: $\zeta = 1.5,\ \omega_n = 2,\ c_0 = 1,\ c_1 = 0,\ s_0 = 0.3,\ s_1 = 0$

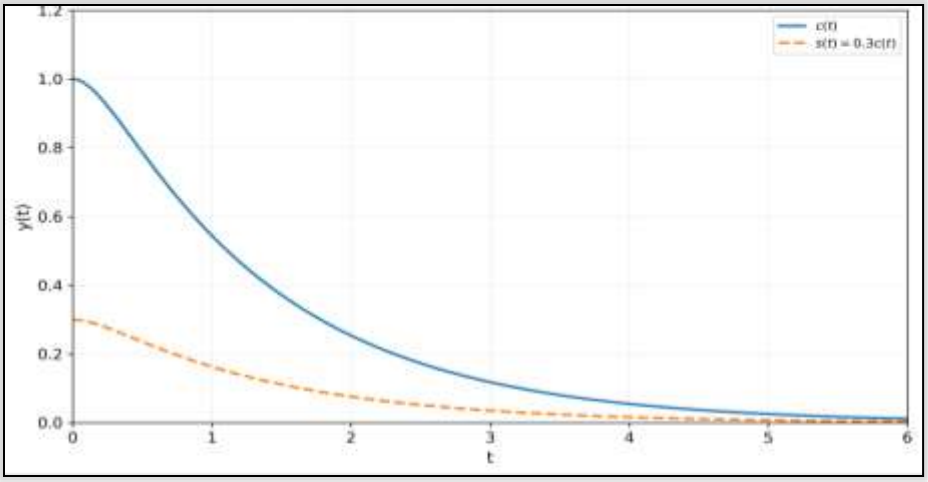

Figure 8B: Over damped oscillator with $\zeta = 1.5,\ \omega_n = 2$.

Remark 8. The spread dynamics follow the same characteristic equation as the center dynamics. Therefore, uncertainty inherits the stability properties of the nominal system, but its initial rate of change determines whether uncertainty temporarily amplifies before decaying.

Example 5: (Time-Varying Coefficient System)

Consider the first-order system with time-dependent coefficient $D_{SGH}\tilde{y}(t) = -t\,\tilde{y}(t)$.

The decomposed system is

$$\begin{cases} c'(t) = -t\,c(t), \\ s'(t) = -t\,s(t). \end{cases}$$

With $\tilde{y}(0) = G(2,0.5)$.

The solutions are

$$c(t) = 2e^{-t^2/2}, \qquad s(t) = 0.5e^{-t^2/2}.$$

This represents a system with super-exponential uncertainty attenuation.

The uncertainty reduction is faster than exponential, demonstrating the framework's ability to describe non-exponential attenuation behaviors. The α-cut is

$$[\tilde{y}(t)]_\alpha = \left[2e^{-t^2/2} - 0.5e^{-t^2/2}\sqrt{-2\ln\alpha},\ 2e^{-t^2/2} + 0.5e^{-t^2/2}\sqrt{-2\ln\alpha}\right].$$

The relative uncertainty level, defined as

$$\rho(t) = \frac{s(t)}{|c(t)|} = \frac{0.5}{2} = 0.25 \quad \text{(constant)},$$

remains constant for all $t > 0$, indicating that although absolute uncertainty decays, the relative uncertainty is preserved.

If alpha-cut ($\alpha = 0.5$): $[\tilde{y}(t)]_{0.5} = [1.4113e^{-t^2/2}, 2.5887e^{-t^2/2}]$

Relative uncertainty: $\rho(t) = \frac{s(t)}{|c(t)|} = \frac{0.5}{2} = 0.25 \quad \text{(constant)}$

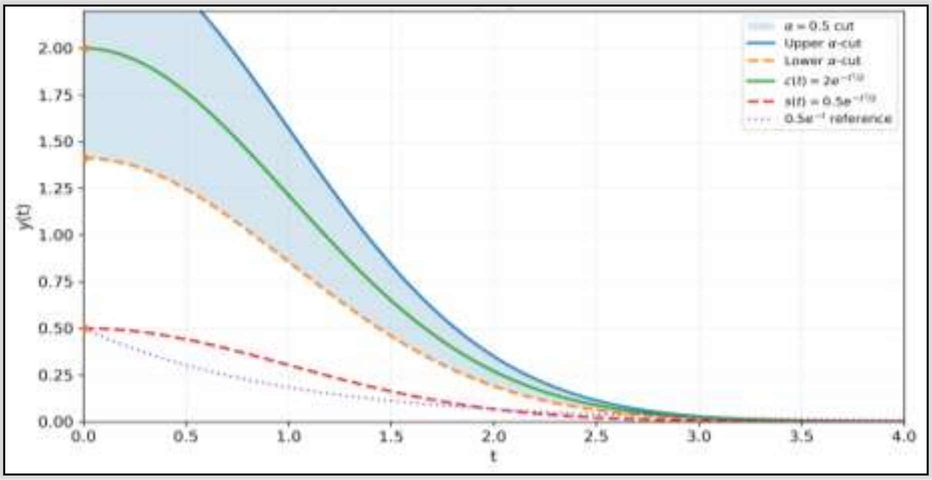

Figure 9: Time-varying coefficient $a(t) = -t$ with super-exponential decay.

Table 4: Summary of All Examples

Example	Type	$c(t)$	$s(t)$	$\rho(t)$
1	Attenuation	$2e^{-t}$	$0.5e^{-t}$	0.25
2	Amplification	$e^{0.5t}$	$0.2e^{0.5t}$	0.2
3	Triangular	$1.5e^{-t}$	e^{-t}	0.667
4	Oscillator	$e^{-0.6t}(\cos + 0.3145\sin)$	$0.3e^{-0.6t}(\cos + 0.3145\sin)$	0.3
5	Super-exp.	$2e^{-t^2/2}$	$0.5e^{-t^2/2}$	0.25

Example 6: Independent Center and Spread Dynamics

Consider the first-order linear system with independent forcing terms: $D_{SGH}\tilde{y}(t) = -\tilde{y}(t) + \tilde{f}(t)$, where $\tilde{f}(t) = G(\sin(t), \cos(t))$ is a Gaussian fuzzy forcing term with independent center and spread forcing. The decomposed system is:

$$c'(t) = -c(t) + sin(t),\ c(0) = 2 \text{ and } s'(t) = -s(t) + cos(t),\ s(0) = 0.5$$

This is a linear system. The solutions are:

$$c(t) = 2e^{-t} + \tfrac{1}{2}(\sin\ t - \cos\ t) + \tfrac{1}{2}e^{-t} \text{ and } s(t) = 0.5e^{-t} + \tfrac{1}{2}(\sin\ t + \cos\ t) - \tfrac{1}{2}e^{-t}$$

Clarification of "Independence": In this context, "independence" does not mean that $s(t)$ is completely decoupled from $c(t)$ in the differential equations. Rather, it means that $s(t)$ is not merely a scaled copy of $c(t)$. In all previous examples, the ratio $\rho(t) = \frac{s(t)}{c(t)}$ was constant, implying that the spread dynamics were completely determined by the center dynamics. In contrast, the presence of the independent forcing term $f_s(t) = \cos(t)$ in the spread equation, while the center is driven by $f_c(t) = \sin(t)$, ensures that $s(t)$ evolves according to its own dynamics. This is confirmed by the fact that $\rho(t)$ is not constant.

Unlike the previous examples, the ratio: $\rho(t) = \frac{s(t)}{c(t)}$ is not constant. Table 5 shows the numerical evolution of $c(t), s(t)$, and $\rho(t)$.

Table 5: Numerical evolution of $c(t), s(t)$, and $\rho(t)$.

t	$c(t)$	$s(t)$	$\rho(t) = s(t)/c(t)$
0.0	2.0000	0.5000	0.2500
0.25	1.4887	0.6646	0.4464
0.5	1.2131	0.8033	0.6622
0.75	0.9941	0.9662	0.9719
1.0	0.7358	1.0919	1.4840
1.25	0.5582	1.2504	2.2401
1.5	0.4463	1.3477	3.0198
1.75	0.3471	1.4477	4.1638
2.0	0.2707	1.5567	5.7507

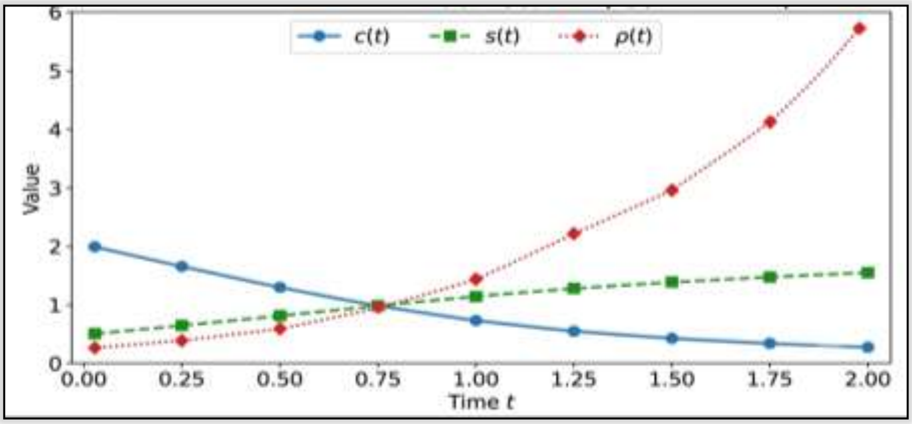

Figure 10: Numerical evolution of $c(t), s(t)$, and $\rho(t)$.

Figure 10 illustrates the temporal evolution of the center $c(t)$, the spread $s(t)$, and the relative uncertainty $\rho(t) = s(t)/c(t)$ for the linear system defined in Example 6. The center $c(t)$ decays monotonically from $c(t) = 2.0$ to $c(t) \approx 0.27$, while the spread $s(t)$ grows monotonically from $s(t) = 0.5$ to $s(t) \approx 1.56$. Consequently, the relative uncertainty $\rho(t)$ increases from 0.25 to approximately 5.75 over the same time interval.

A key feature of this figure is the intersection point where $c(t)$ and $s(t)$) cross, which occurs at $t \approx 0.84$. At this instant, we have $c(t) = s(t) \approx 0.94$, which corresponds to $\rho(t) = 1.0$. This point marks a qualitative transition in the system behavior: for $t < 0.84$, the uncertainty is smaller than the nominal signal $s < c$, indicating that the system remains relatively predictable. For $t > 0.84$, the uncertainty dominates the nominal signal $s > c$), meaning that the fuzziness becomes the primary characteristic of the state.

Unlike the previous examples (1–5), where the ratio $\rho(t)$ remained constant, the variation of $\rho(t)$) in Figure 6 confirms that the spread dynamics are not merely a scaled replica of the center dynamics. This validates the claim that the spread variable $s(t)$ possesses its own independent dynamics, driven by the independent forcing term $f_s(t) = \cos(t)$), while the center is driven by $f_c(t) = \sin(t)$. This separation is the core novelty of the proposed center-spread framework, as it allows for explicit analysis of uncertainty propagation independent of the nominal trajectory.

8 Conclusion and Future Work

In this paper, a dynamic center-spread framework has been proposed for a class of linear fuzzy differential systems under strongly generalized Hukuhara differentiability. The main idea of the proposed approach is to introduce the uncertainty magnitude as an independent dynamic variable coupled with the nominal system evolution. The uncertain state is represented as $\tilde{y}(t) = G(c(t), s(t))$, where $c(t)$ describes the center trajectory and $s(t)$ describes the uncertainty propagation. The original fuzzy differential problem has been transformed into two interacting deterministic components: a center equation and a spread equation. This decomposition provides a direct interpretation of the evolution of uncertainty and allows the study of uncertainty attenuation and amplification separately from the system dynamics. An existence and uniqueness theorem has been established for admissible center-spread solutions under suitable continuity and Lipschitz conditions. Furthermore, a center-spread Laplace representation has been introduced by applying the classical Laplace transform independently to the center and spread variables. The consistency property has been proved by showing that the proposed framework reduces to the classical deterministic model when $s(t) = 0$. Analytical examples demonstrated that the spread component can either decrease or increase according to its evolution law, providing a clear description of uncertainty behavior. The proposed framework is developed for Gaussian-type fuzzy numbers, where uncertainty is fully characterized by a single spread parameter. This choice allows for a clean decomposition into center and spread dynamics while maintaining analytical tractability. The proposed framework does not replace the general theory of fuzzy differential equations; rather, it provides a structured model for a class of uncertain systems where the uncertainty magnitude itself has a meaningful dynamic interpretation.

Future work will focus on the following directions:

1. Non-symmetric fuzzy numbers: Extending the framework to handle two spread parameters (left and right spreads) to accommodate non-symmetric uncertainty profiles, with a generalized reconstruction operator $\tilde{y}(t) = G(c(t), s_L(t), s_L(t))$, and revised admissibility conditions.
2. Nonlinear systems: Developing numerical methods and analytical techniques for the coupled nonlinear case where the center and spread equations have different structures, as indicated in equation (11).
3. Fractional fuzzy differential systems: Applying the center–spread decomposition to fractional-order fuzzy differential equations, including fractional admissibility conditions and fractional Laplace representations.
4. Control theory: Using the framework for the analysis and design of fuzzy control systems, including stability and controllability studies with uncertainty-related cost functionals.
5. Numerical algorithms: Developing efficient numerical algorithms for large-scale applications, including adaptive step-size methods, error analysis, and parallel implementations.
6. Experimental validation: Applying the proposed framework to real-world problems in engineering, physics, and finance where uncertainty propagation is critical, and validating the extraction algorithms on experimental data.

Note 2: Addressing these directions will establish the center–spread dynamic approach as a versatile paradigm for uncertainty quantification in fuzzy differential systems.

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